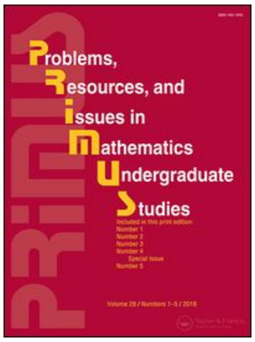

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The Calculus Mastery Exam: A Report on the Use of Gateway-Inspired Assessment Tools at Liberal Arts Colleges

Eric Barth and Ryan S. Higginbottom

ABSTRACT
Gateway testing is an important pedagogic tool employed by many university mathematics departments in calculus and pre-calculus courses. With a goal of ensuring that students attain needed basic skills in courses with a conceptual "reform" orientation, these tests provide an efficient means of assessing a large volume of student work, while at the same time motivating students to engage in the needed skills-based practice. In this article, we describe our experience over the past decade in utilizing a version of this idea, which we call *Mastery Exams*, in the context of small liberal arts institutions. We discuss students' experience of autonomous mastery goal attainment inherent to our approach in the context of Self Determination Theory applied by educational psychology researchers in a number of higher education settings. We describe the implementation of these exams in the small-class-size setting and present data from a small study carried out at Washington & Jefferson College in 2012. We found that students in the Mastery Exam group showed significant gains in a pre/post-test of nonroutine calculus tasks compared to the non Mastery Exam group.

KEYWORDS
Calculus; mastery exams; gateway tests

1. INTRODUCTION

In mathematics education, the relationship between students' conceptual understanding and mastery of basic skills is variously understood as complementary or competing, and the proper balance is hard to strike [16, 22].

It has been three decades since the watershed Tulane Conference in 1986 that launched the "reform" movement in undergraduate calculus. Since that time mathematics educators have taught and designed courses with the age-old aim of finding the right concepts/skills balance, mindful of the reformers' critique of traditional courses that Ronald Douglas summarized at the time as "... many of those who do finish the course have taken a watered down, cookbook course in which all they learn are recipes, without even being taught what it is that they are cooking [5]."

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Gavin LaRose describes the right balance this way:

> However, while these skills are intrinsic to or essential for success in the courses, they are not our educational focus. Our courses focus on conceptual understanding; e.g., for Calculus I, what a derivative tells us, and how we can use this to solve problems – not how to find the derivative of $\sin(\cos(x^2) + 2)$. This conceptual understanding is fundamental to our courses, and it is therefore the subject of the bulk of class and instructor time, and it is what we evaluate on exams. Gateway exams are the mechanism by which we ensure that students also have or acquire the basic skills we expect of them, and we find that they work with minimal investment of time in-class [10].

Gateway testing has been used as an important part of Calculus instruction at universities for many years [6, 14]. LaRose and Megginson write: "Gateway tests provide a means of assuring that students in reformed precalculus and calculus courses acquire the algebraic and computational skills needed in courses following these, while allowing the focus of the course to be on the conceptual understanding intrinsic to the reformed courses [11]." In the large university setting, " . . . logistical difficulties plague the pencil-and-paper administration of these tests, significantly decreasing their usefulness [11]." Online implementations with automatic grading allow gateway tests to be efficiently administered at institutions with large calculus enrollments. (LaRose and Megginson report single-semester Calculus I registrations of over 1500 students.)

Gateway testing provides a measure of quality control over students' acquisition of traditional technical skills that are assumed in subsequent courses in science and mathematics. As such, gateway testing provides summative assessment instruments for the purposes of external accountability. At the same time, it is generally believed that the high standard for passing, together with the possibility of multiple attempts for each student to attain that standard, contribute to the success of the gateway test as an instrument of formative assessment. Over the past 25 years, the gateway testing concept has become an established part of rigorous reform calculus curricula [2].

At small undergraduate institutions the logistical difficulties described above are less pronounced. We are convinced of the importance of developing student skills as a prerequisite for deep and meaningful conceptual understanding and problem-solving ability in calculus. In our view the gateway testing idea with its high performance standards and opportunity for student practice can combine very effectively with the fundamental opportunities for frequent and meaningful student-instructor interaction upon which smaller institutions are founded. Beginning at Kalamazoo College in 1998, and later at Washington & Jefferson College, we have incorporated the high-standards, multiple-attempt strategy in a format that we believe fits the liberal arts context. In this article we describe our rationale in developing this idea, which we call The Mastery Exam, including the student outcomes we target in their use. In addition to the role this experience plays in coverage of important subject material, we employ the framework of Self Determination Theory to explore this intentional focus on self-paced and intrinsically-motivated mastery goal setting as it shapes the students' experience of the course in positive ways. We present the implementation details we have developed over the years, and we report on a small study of student performance outcomes as measured by pre- and post-testing in two calculus sections at Washington & Jefferson College.

2. GATEWAYS AND MASTERY: DESIRED STUDENT OUTCOMES

In order for students to meaningfully engage the beautiful concepts of calculus, we are convinced that they need a mastery of computational skills: derivative rules, integration techniques, infinite series methods, etc. Our goal in this work is to create space in our calculus courses for conceptual exploration and applications of calculus methods in a rich variety of settings without de-emphasizing mastery of basic skills.

We begin by asking, “What is Mastery?” We say that a student has obtained mastery of a set of skills when the student can, without outside help, respond to a challenge (in our case, a set of exercises chosen to represent the skills in question) without error and with confidence. Mastery is a key aspect of achievement goal theory as applied to human motivation in teaching and learning [13]. In experiencing mastery, not only has the student completed a task without error, but also the student will have learned to identify questions or uncertainties and will be motivated to confidently address them. Our emphasis on high performance and cultivation of intrinsic motivation has been central to this approach from the start. We intend that our definition of mastery overcomes many of the extrinsic motivation features of high-stakes assessments for which levels of achievement correspond to varying degrees of partial correctness and incomplete understanding. Further, our implementation of mastery assessment actively encourages students to engage with their instructors in ways that develop relationships built on working together to achieve well-defined goals with objective success not subject to notions of partial credit, curved grades, or other features of traditional mathematics classrooms that rely upon and emphasize power differential and control.

This definition suggests that, in addition to the mathematical skills that we hope students will acquire and display through this process, our thinking has also been formed by the possibility that students will additionally gain meta-cognitive skills and instrinsic motivation that will allow them to confidently face future academic challenges. We have recently become aware of an influential theory of motivation – Self Determination Theory (SDT) [17] – that neatly identifies three needs or conditions for human motivation: Competence, Autonomy, and Relatedness. Self Determination Theory has been used as a helpful construct in understanding student motivation in the mathematics classroom [4, 12, 18] and other post-secondary instructional settings such as engineering [19] and physician training programs [9].

Cerasoli and Ford have recently shown, [3], using a study of students in a university psychology class, that the acquisition of mastery goals such as those we describe here explains the relationship between students’ intrinsic motivation and performance. In fact, the emphasis on mastery goals catalyzes an increase in students’ intrinsic motivation.

In addition, mastery-based assessment has for many years played an important role in the discussion of increasing inclusion in the STEM classroom [8]. The mastery learning philosophy asserts that " . . . under appropriate instructional conditions, virtually all students can and will learn well [1] . . . " As our institutions address anew questions of inclusiveness and structural inequality in education, our mastery exam process provides a leading example across the STEM fields of pedagogical practices that go beyond traditional notions of student preparedness and that are informed by an understanding of human emotion and motivation as it relates to learning and academic performance.

We state the desired student outcomes of our mastery exam process using this SDT construct:

- Students will demonstrate mastery of traditional algebra and calculus skills through mastery goal-oriented practice
- Students will demonstrate clear writing in the style of the discipline
- Students will demonstrate improved understanding of calculus concepts in "nonroutine" applied problems
- Students will demonstrate an increased sense of confidence and self motivation for competence in algebra and calculus skills
- Students will develop increased metacognitive awareness, including increased use of autonomous error-focused practice as an effective tool for academic achievement
- Students will develop an increased sense of relatedness to the instructor through working together to meet shared mastery goals

3. IMPLEMENTATION OF MASTERY EXAMS IN THE SMALL UNDERGRADUATE INSTITUTION

From the beginning, our aim has been to make use of technology in ways that support the student–instructor relationship, while avoiding uses of technology that replace or circumvent direct faculty–student interaction, even when such technologies might expedite course management. For example, we find web-based homework and quiz resources helpful in that they can provide automatic feedback to students as they begin to work on new material. We generally incorporate this activity into our classes as a precursor to meaningful classroom engagement and as preparation for students' carefully written work on traditional homework and quizzes. These latter assessment instruments receive instructors' thoughtful reading and feedback. In this way, technology can provide a context for richer experience upon which relational experiences in and out of the classroom can be built more fully than in the past. We developed the web-based component of the mastery exam process to exploit the opportunities that a wide variety of example problems affords our students to explore, discover and identify areas of confusion. This rich experience helps students learn to identify autonomously what they can and cannot (yet) do, and builds intrinsic motivation to address these identified challenges. The face-to-face student–instructor components are intentionally focused on building stronger learning relationships through shared mastery goal oriented practice and low-stakes opportunities to demonstrate competence.

The instructors in our departments have decided on the specific skills that are to comprise the students' mastery exam experiences. For example, the Differentiation Skills Mastery Exam included in Appendix 1 reflects eight desired differentiation skills: elementary functions, products, quotients, chain rule, hybrid products and quotients, hybrid composite functions, logarithmic differentiation, implicit differentiation. Within that framework we have implemented an online system of mastery exams that provides students with a nearly endless supply of examples to work on, with each type of problem being drawn at random from a pre-generated list of a dozen or more examples. We note that while our bank of examples includes instances of all desired function types (trigonometric functions, logarithms, exponentials, rational functions, etc) we have not implemented the random selection of problems in a way that guarantees that every desired type of function will appear on every test – the randomly generated exam in the appendix contains two instances of the cosine function and none of tangent function for example. Rather, we content ourselves that preparation for and practice with the randomly generated problems ensures adequate coverage for our students. Details of the software implementation are presented at the end of this section.

We typically carry out a Mastery Exam experience two or three times in each term of Calculus. At the beginning of the first course, we administer an Algebra Skills Mastery Exam. At the end of the term, students take a Differentiation Skills Mastery Exam. At Washington & Jefferson College, there is an additional mid-term Limit Skills Mastery Exam. The overall Mastery experience contributes from 10 to 15 percent of the student's course grade. It is worth noting here that the initial mastery experience at the beginning of the semester serves our aim of fostering student relatedness in a number of ways:

- Students and instructors work together face-to-face right from the start of the term.
- Students and instructors alike are able to discover what mathematics skills students have and have not (yet) mastered, which can have a direct influence on instructors' choices for most effective classroom instruction.
- Students become acquainted from the start with the sometimes unfamiliar expectation of careful and complete written communication of their mathematics work.

3.1. What We Ask Students to Do

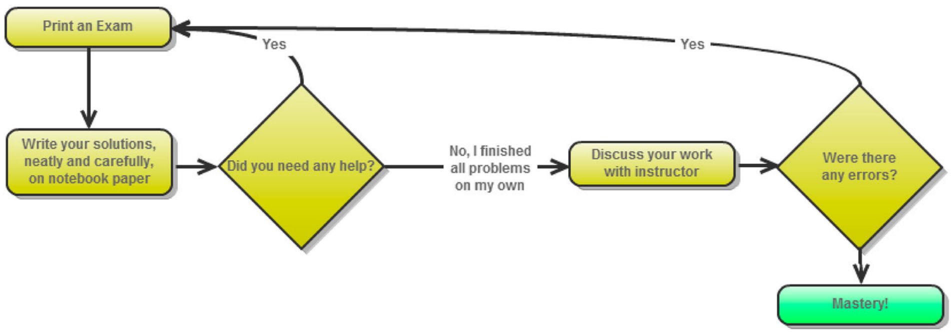
In our experience, the key to meaningful student engagement in the mastery exam experience is clear and frequent discussion between students and instructors of the department's expectations, always informed by the student outcomes listed above. Especially at the beginning of a semester, this builds the sense of relatedness between instructor and student. The central question, "What is mastery?" anchors the discussion. We ask that students complete the following steps:

(1) Access the mastery web site and print (or copy by hand) the page of problems that appear there.
(2) Write solutions neatly and carefully. We emphasize that this is not simply an exercise in correct mathematical manipulation. Equally important for authentic sense of student competence is to demonstrate an ability to communicate the solution process to a reader in the accepted style of the discipline. This marks a key departure from the traditional gateway testing, in which student responses (either as multiple choice or increasingly as student-entered mathematical expressions) are automatically logged by the web server and reported to instructors. We believe that the free response format directly addresses a number of our instructional goals.

    We state emphatically and sympathetically in advance that some of the problems students will encounter on the mastery exams are very difficult – students will almost certainly encounter problems about which they are unsure, especially in the early stages of the mastery experience. Students are encouraged to plan and execute their work autonomously and to seek help in any form available: classmates, instructors, tutors, books, notes, online sources, etc. However, once a student has made use of any outside help, they have not yet displayed mastery. Once all of the difficulties on a given exam have been addressed through outside help, the student begins again at step (1).
(3) When the student has written solutions to all the problems on a mastery exam and feels confident that all the work is correct, the student makes an appointment to discuss the exam with the instructor. These appointments typically require about 10 minutes. To schedule these appointments, we have had success both using paper sign-up sheets that are passed around in class, as well as online sign-up services that allow students to add, delete, and change appointments at any time. We have found that for a class of N students, between $1.5N$ and $2N$ appointment slots are needed: some students present perfect papers at their first appointment, while a number are helped by the instructor to see errors they had overlooked, leading to step (1), and eventually to another discussion in the instructor's office.

    It is important to note that the commitment of instructor's office time is considerable. For a class of 30 students, the rule of thumb calls for up to 600 minutes – 10 hours – of office time. During the course of a week we often schedule these appointments during existing office hours, which ameliorates the requirement of additional blocks of time. The question for any instructor considering this approach is "Does the extra student contact time provide positive student outcomes that balance the obvious personal demands this places on my work week?" We believe the answer is a resounding "Yes!"

Our course syllabi condense the process into a flowchart:



3.2. How Do We Know the Students Won't Cheat?

Central to the concerns that led to the calculus reform movement thirty years ago were observations of students' dissatisfaction: feelings of boredom, lack of appropriate challenge, and disconnectedness with the underlying ideas and purpose of the course. Researchers in educational psychology and Self Determination Theory link these kinds of student frustrations to educational environments that students perceive as strictly controlled by the instructor to the extent that students' fundamental needs for autonomy, competence and relatedness are frustrated. The result is maladaptive motivational functioning on the part of the students such as anxiety, pressure, internal conflict, and cheating in order to achieve an external reward (in this case the points for this portion of the course grade) [7]. It is interesting to read the recommendations of the MAA report from the Tulane conference [5] through the lens of Self Determination Theory, as many of the underlying calculus reform objectives align with an implicit acknowledgment of the universal elements of human motivation on the part of both students and instructors.

The Mastery Exam process described above puts a great deal of responsibility into the students' hands, returning autonomy over an important aspect of the course. Students are encouraged to work on their Mastery Exam tasks whenever and wherever they like, at their own pace within a reasonable time frame. Admittedly there are many opportunities to present the work of others as one's own. Over the years we have operated under the assumption that academic dishonesty is not a significant problem, consistent with the findings of Cerasoli and Ford [3]. We are confident that students' opportunities to experience autonomous mastery and to share that sense of competence with others is a powerful positive motivator. We reinforce that belief with several practical steps that our instructors – even those who are new to the mastery exam process – have found workable and easily implemented:

(1) Frequent classroom conversations, including illustrative scenarios, built around the definition we gave above in answer to the question "What is mastery?" Presented with good humor and acknowledgment that "we've all been there" when it comes to the temptation to give less than our best, we find that students are quick to identify unfairness and call attention to it. This begins to create the needed ethic of self-motivated mastery.
   - Suppose you're working on a Mastery Exam and get stuck. You ask another student for help. "Is this mastery?"
   - Suppose you can't remember how to do implicit differentiation while you're working on your Mastery Exam, so you look it up online. "Is this mastery?"
   - You get stuck on a problem, so you print another Mastery Exam in hopes that you get an easier selection of problems. "Is this mastery?"

(2) We emphasize the low-stakes nature of any single attempt at the Mastery Exam. While it is true that Mastery entails error-free work, it comes from error-focused practice – putting time and effort into the things that need it. The Mastery Exam experience is, in our minds, the most efficient means of developing the habit of error-focused practice.

(3) The Student-Instructor discussion of the student's work gives the experienced instructor great insight into the veracity of the student's claim of independent work. When asked about instances of student dishonesty in mastery exams, all the instructors in our departments reported that "Walk me through this" is a prompt that reveals the student's level of understanding and ownership of the work presented. In this low-stakes context, a stern rebuke or formal action for academic dishonesty hasn't ever been needed. Instead, students respond well to a gentle "I'd like to see another exam from you just to make sure you've really attained mastery." We bring this aspect of the student-instructor discussion to the students' attention from the beginning.

(4) At Kalamazoo College we have a strong tradition of empowering students to abide by the principles of an Honor System set out in The Student Code of Conduct. The relevant section is labeled "Nurturing Independent Thought":

   > To safeguard the integrity of academic work and research, we accept responsibility for our own scholarly performance. We regard false representation of our scholarly work as unacceptable because it undermines our integrity and that of the community. We commit ourselves to knowing under what conditions scholarly research is to be conducted, the degree of collaboration allowed, and the resources to be consulted.

   A similar code of conduct is in force at Washington & Jefferson College.

(5) Finally, we suspect that there may be a few students who will falsely represent work that is not their own and avoid all detection. We weigh this against the self-motivation that can be gained by the students who approach the task honestly. It is our belief that the framework described here, founded on self-regulation by individual students, brings about a sense of autonomy and

competence not possible under strictly-controlled regimes obviously designed primarily to thwart cheaters.

3.3. Software Implementation Details

At both schools, we generate each individual mastery exam from a large collection of problems with an online system developed internally that randomly selects problems by topic. As more widely and easily implementable alternatives, we have also created prototype mastery exams using the random quiz capabilities of the course management platforms Moodle [15] and Webassign [20]. Others [10] have implemented Gateway testing in *WeBWorK* [21].

4. MEASURING STUDENT OUTCOMES

In the spring semester of 2012, two sections of Calculus I were taught by two different instructors at Washington & Jefferson College. The syllabus of one section included the Mastery Exam experience (ME) as described above ($N = 19$) and the syllabus of the other section (NME) did not ($N = 20$).

The ME class met Monday, Wednesday, and Friday from 11:45–12:50, while the NME class met on the same days from 1:00–2:05. Both courses included two midsemester exams, a final exam, quizzes, assignments, and occasional graded homework presentations. While the textbook and final exam were common between the two sections of the class, the other assessments were created by the instructors without collaboration.

In keeping with our central idea that the Mastery Exam experience facilitates the complementary relationship between skill building and conceptual understanding, we focused on the student outcome of increased calculus understanding for concepts in nonroutine applied problems. To measure this improvement, we administered a pre-test at the beginning of the term which focused on the pre-calculus functional representation aspect of applied problems that students would encounter later in the term. For example, on the pre-test a student would be asked to write a function for the area of a planar region subject to some geometric constraints. Students would then encounter on the post-test portion of the final exam the corresponding applied calculus problem: find the dimensions of the region (subject to the constraints) that minimize the area.

4.1. Do Mastery Exam Students Perform Better on Non-Routine Calculus Tasks?

We sought to determine whether the Mastery Exam experience led to greater gain on the nonroutine applied problems on the final exam over the pre-test, compared to the NME group. The difference between the ME and NME groups, as percentage-point gains from the pre-test to the post-test, was used to measure the effectiveness

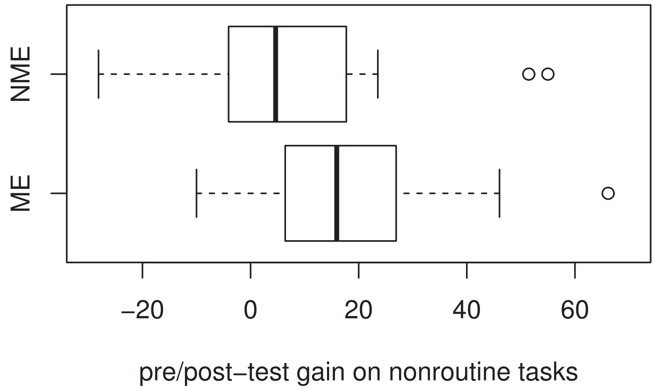


Figure 1. Pre/post-test gain for each of the sections: mastery exam experience (ME) and no mastery exam experience (NME).

Table 1. Five number summaries of pre/post-test gain on nonroutine tasks.

	Min	Q1	Median	Q3	Max
All groups	−28.1	1.2	11.4	22.0	66.1
ME group	−10.0	6.4	15.9	26.9	66.1
NME group	−28.1	−4.1	4.6	17.7	55.0
No calculus	−11.5	6.4	12.4	21.3	37.6
Some calculus	−28.0	−2.5	6.4	37.2	66.1
AP calculus	−28.1	−1.7	5.3	17.9	51.5

of the ME experience on this learning outcome. Summary data for this measure are presented in Table 1 and Figure 1. The mean (SD) gain in the ME group was 18.5 (18.6), one-tail 95% CI $[11.1, \infty]$, $p = 0.0002$, compared to 6.9 (21.4), 95% CI $[-1.3, \infty]$, $p = 0.08$ for the NME group. At the $\alpha = 0.05$ significance level, only the ME group showed significant gain.

The mean difference in improvement between the ME and NME groups was significant at the $\alpha = 0.05$ level, with mean gain difference 11.6, one-tail 95% CI $[0.8, \infty]$, $p = 0.04$.

4.2. Do Mastery Exam Students Perform Better on Routine Calculus Tasks?

Our foundational idea in developing the mastery exam experience was that a heightened sense of confidence in routine calculus computations would allow students to focus and learn more fully in nonroutine contexts involving problem solving and decoding math content from verbal descriptions. Our observation was that this sense of mastery was not being attained from traditional assessment methods, even though students in the traditional setting seemed to score well enough in routine tasks on traditional assessment instruments like in-class tests.

We compared student performance on the routine basic skills portion of the final exam. Scores on the routine section of the final exam are summarized by group in Table 2 and Figure 2. For routine computations, as measured by the relevant section

Table 2. Five number summaries of final exam scores on the routine computation.

	Min	Q1	Median	Q3	Max
All groups	41.2	69.1	76.5	91.2	97.1
ME group	42.6	72.8	85.3	91.2	97.1
NME group	41.2	63.2	74.3	91.2	97.1
No calculus	69.1	74.3	87.5	91.2	95.6
Some calculus	60.3	70.6	80.1	95.6	97.1
AP calculus	41.2	55.9	66.9	83.8	92.7

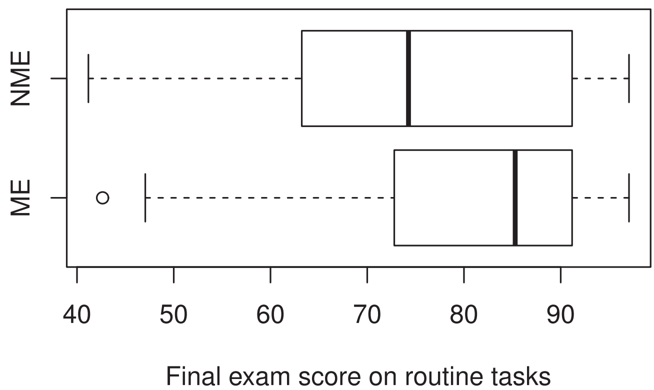


Figure 2. Final exam scores on the routine computation in each of the sections: mastery exam experience (ME) and no mastery exam experience (NME).

of the final exam, although mean and median scores are higher in the ME group there was not a significant difference between the groups, likely due to the high degree of variability in each group. Mean (SD) score for the ME group was 79.3 (16.2) compared to 75.0 (15.1) in the NME group, with mean difference 4.3, one-tail 95% CI $[-4.2, \infty]$, $p = 0.20$.

4.3. Are these Results Related to Level of Student Preparation Prior to the Course?

Suspecting that those who were seeing Calculus concepts for the first time might be differently affected by the Mastery Exam experience, we surveyed the students about their level of mathematics preparation prior to registration in the course, classifying their background as "has taken AP calculus in high school," "has taken non-AP calculus in high school," and "no calculus in high school." Note that we did not collect information about students' grades in those high school courses, nor any information about scores on subsequent AP exams. Counts for each category are given in Table 3. Chi-square analysis showed no significant association between preparation level and group ($p = 0.29$). Within each group, ANOVA showed no significant association between preparation level and pre/post-test gain ($p = 0.43$ for ME and $p = 0.94$ for NME). Figure 3 summarizes the pre/post-test gains within each category for each of the two mastery groups.

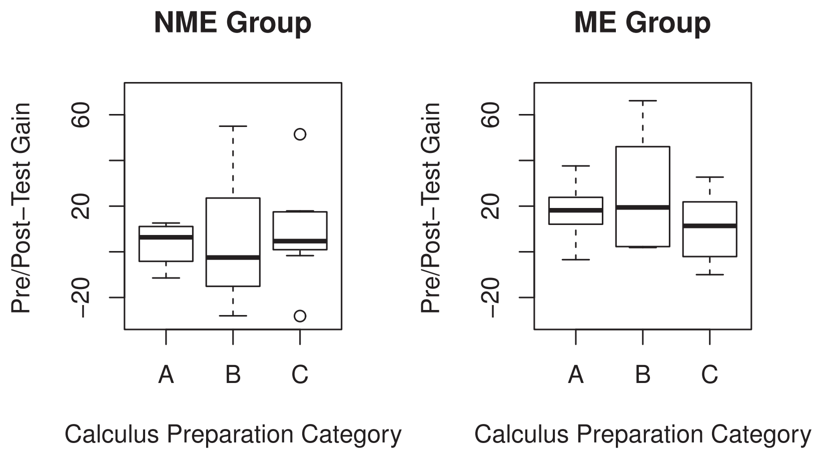


Figure 3. Pre/post-test gain for each of the sections vs calculus background: A no previous calculus, B some calculus, C AP calculus. Within each class (ME and NME) there is no obvious pattern of association between preparation level pre/post-test gain.

Table 3. Distribution of calculus backgrounds in the ME and NME groups.

	ME	NME
No calculus	8	4
Some calculus	6	6
AP calculus	5	9

Note: One student did not report on the question of calculus background.

5. CONCLUSIONS

We have described the rationale and implementation of mastery exams – an adaptation of the gateway testing process commonly used at large universities – that we believe serves well the pedagogical goals of mathematics departments at small undergraduate institutions. The mastery exam experience for students aligns nicely with Self Determination Theory, a powerful theoretical model of human motivation. Through these low-stakes, high-standards assessments students experience an opportunity to autonomously attain goals and display competence in important mathematics skills. Web-based automation allows for a rich variety of example problems for exploration, discovery, and identifying areas of confusion. The face-to-face student–instructor component of the mastery exam experience builds the learning relationship around shared mastery goals, which in turn fosters greater student self-motivation.

We report results of a very small study which suggests that student learning and performance on non-standard conceptual tasks in undergraduate calculus classes is enhanced by the mastery exam experience. The study is limited by the small sample sizes, the data variability within each group, and the fact that important factors such as instructor and specific assessment materials were not controlled between the groups.

ACKNOWLEDGEMENTS

The authors thank Dr. Faun Doherty (Washington & Jefferson College) for allowing them to use her Calculus I class as a control group for this study.

APPENDICES

APPENDIX 1. SAMPLE DIFFERENTIATION SKILLS MASTERY EXAM

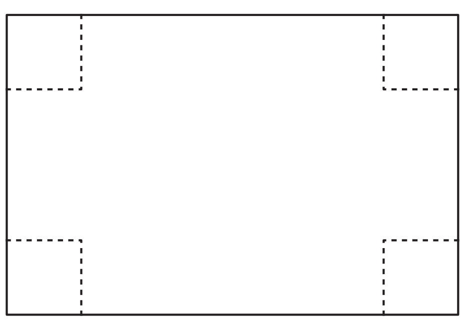
Find the derivative for each of the following functions.

(1) $f(x) = \pi x^2 - \frac{5}{x^3} + e^x$
(2) $f(x) = (1 - x^2)\cos(x)$
(3) $f(y) = \frac{\sin(y)}{y}$
(4) $h(t) = \sin(t^2 + 2)$
(5) $w(t) = \frac{t^2-49}{\sqrt{t+7}}$
(6) $h(r) = 4\cos^7(2 - 4r)$
(7) $h(x) = (3x + 1)^x$
(8) Find $\frac{\mathrm{d}y}{\mathrm{d}x}$: $\ln(x - y) = 4y^2 + x$

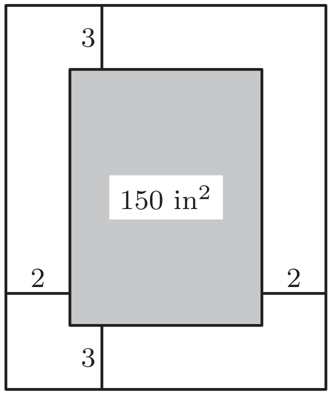
APPENDIX 2. PRE-TEST

Students were given a total of 20 minutes to complete these problems.

(1) Suppose that a function f is defined by the formula $f(x) = 3x^2$. In complete sentences, explain the difference between the symbols $f(x)$ and $f(3)$.
(2) A rectangular box is to be constructed by cutting congruent squares out of the corners of an aluminum sheet and folding up the sides. If the piece of aluminum with which you begin is 12 feet by 8 feet, write down a function which measures the volume of the resulting box. This function should have only one independent variable.



(3) We want to make a copper plaque containing 150 in^2 of writing area. We need 3 inch margins on the top and bottom, and 2 inch margins on the sides. (See diagram below – the writing area is contained within the margins.) Introduce appropriate notation and write down a function which measures the area of the entire plaque. This function should have only one independent variable.



(4) Write down the coordinates of the triangular region bounded by the graphs of the curves $y = x^2$, $x = 1$, and $y = 4$. (Note that this region is not perfectly triangular, but it is triangular-ish.)

APPENDIX 3. FINAL EXAM

This is the final exam. Problems 1, 7, and 11 were labeled as routine problems; the rest were classified as nonroutine.

(1) (12 points) Evaluate the following limits, if they exist. If a limit does not exist, explain in detail why it does not exist.
 (a) $\lim_{x\to-5} \frac{x^2+3x-10}{x^2-x-30}$
 (b) $\lim_{x\to\pi} \frac{2x-1}{\sin(x)}$
 (c) $\lim_{x\to0} f(x)$ if $f(x) = \begin{cases} 4-x & x<0 \\ 3 & x=0 \\ \cos(x) & 0<x<1 \\ x^2+1 & x\geq 1 \end{cases}$

(2) (10 points) Consider the following function.

$$f(x) = \begin{cases} x-7 & x<0 \\ x^2+ax+1 & 0\leq x<1 \\ 7-\sqrt{3+x} & x\geq 1 \end{cases}$$

 (a) Is there a value of a which will make this function continuous at $x = 0$? If so, find it. If not, explain your answer in detail.
 (b) Is there a value of a which will make this function continuous at $x = 1$? If so, find it. If not, explain your answer in detail.

(3) (10 points) Consider the curve defined by the following equation.

$$2 + x^2y^2 = \sin(y) - 2x$$

 (a) Find the point on the curve which corresponds to the y-value $y = 0$.
 (b) Calculate the equation of the tangent line to the curve at this point.

(4) (10 points) Using the *limit definition of the derivative* (this means no shortcuts!) find the derivative of the following function:

$$f(x) = \frac{3}{2x}.$$

(5) (10 points) Jane has a giant ice cream cone (circular cone with volume $V = \frac{1}{3}\pi r^2 h$). It is 10 cm high and 5 cm in radius at the top. Her ice cream is melting rapidly into the cone at a rate of 2 cm^3/min. How fast is the ice cream level in the cone rising when the melted ice cream in the bottom of the cone is 6 cm deep?

(6) (14 points) Consider the function $f(x) = \frac{2x^2+3x+13}{x-1}$. (Note: to give you plenty of room, this problem continues on the next page.)

- (a) On what intervals is the graph of f increasing? On what intervals is the graph of f decreasing? Explain your answers.
- (b) Find and identify any relative extrema of f. Explain how you identified each point.
- (c) On what intervals is the graph of f concave up? On what intervals is the graph of f concave down? Explain your answers.
- (d) Find any inflection points of f. Explain why each point is an inflection point.

(7) (12 points) For each of the following functions, find the derivative. Do not simplify.

- (a) $f(x) = \frac{x-\sin(x)}{2x^2+1}$
- (b) $f(\theta) = \sqrt[3]{2\theta^2 - \theta + 1}$
- (c) $f(x) = \sqrt{x}(\cos^4(x))$

(8) (4 points) Essay: What is the difference between $\int 3x^2\,dx$ and $\int_2^3 3x^2\,dx$? Your answer should include how these differ in terms of their definition and the way we evaluate them. (Do *not* evaluate the integrals.)

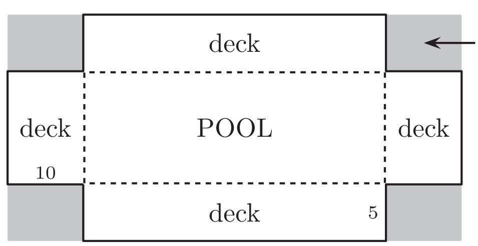
(9) (10 points) Using the *definition of the definite integral* (this means no shortcuts!) and the formulas listed below, calculate the following:

$$\int_1^3 (x^2 + 1)\,dx.$$

$$\sum_{i=1}^{n} c = cn \qquad \sum_{i=1}^{n} i = \frac{n(n+1)}{2}$$

$$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6} \qquad \sum_{i=1}^{n} i^3 = \left[\frac{n(n+1)}{2}\right]^2$$

(10) (10 points) A rectangular swimming pool is to be built with an area of 1800 square feet. The owner wants 5-foot wide decks along either side and 10-foot wide decks at the two ends of the pool. Find the dimensions of the pool that will require the smallest area of rectangular piece of property on which the pool and decks can be built satisfying these conditions. (Please ask your instructor if you have questions about the wording.)



(11) (10 points) Evaluate the following integrals.

- (a) $\int \sin(5\theta + 1)\,d\theta$

(b) $\int_0^1 \frac{y}{(y^2+4)^4}\,\mathrm{d}y$

(12) (6 points) Calculate the area of the region bounded by $y = x^2$ and $x = y^2$.

(13) (12 points) Let R denote the region bounded by $y = \sqrt{x}$, $y = 2$ and $x = 0$.

(a) Form the solid S by rotating R around the x-axis. Set up, but *do not evaluate*, an expression which calculates the volume of S.

(b) Form the solid S by rotating R around the y-axis. Set up, but *do not evaluate*, an expression which calculates the volume of S.

(c) Form the solid S by rotating R around the line $y = 3$. Set up, but *do not evaluate*, an expression which calculates the volume of S.

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