Calculus 3 Sample Final Exam
May 30, 2018

1. **True or False? Explain.**
   (a) The line given by $x = 1+t, y = -2t, z = 3-t$ is perpendicular to the plane $-x+2y+z = -6$.
   (b) If C is a differentiable curve whose tangent vectors are orthogonal to a continuous vector field $\mathbf{F}$ at each point on C, then $\int_C \mathbf{F} \cdot d\mathbf{r} = 0$.
   (c) If $\mathbf{F}$ is a conservative vector field, then $\int_C \mathbf{F} \cdot d\mathbf{r} = 0$ for all curves C.
   (d) For the function $f(x, y) = x^2y - y^2x$, the maximum rate of change at the point $(1, 1)$ occurs in the direction $\langle 1, -1\rangle$.
   (e) The curvature at any point on the circle $x^2 + y^2 = r^2$ is $\kappa = 1/r^2$.
   (f) A conservative vector field $\mathbf{F}(x, y) = \langle P(x, y), Q(x, y)\rangle$ is defined as one for which $\dfrac{\partial P}{\partial y} = \dfrac{\partial Q}{\partial x}$.
   (g) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{xy}{x^2 + y^2} = 0$

2. **Optimization.** A rectangle with length L and width W is cut into four smaller rectangles by two lines parallel to the sides. Find the maximum and minimum values of the sum of the squares of the areas of the smaller rectangles in two ways:
   (a) Optimize a function of two variables x and y that take into account the dimensions L and W.
   (b) Use Lagrange multipliers to optimize a function of four variables subject to two constraints involving L and W.

3. **Differentiability and partial derivatives.** Consider the function defined by

$$f(x, y) = \begin{cases} \dfrac{xy}{x^2 + y^2} & \text{for } (x, y) \neq (0, 0) \\ 0 & \text{for } (x, y) = (0, 0) \end{cases}.$$

   (a) Show that f is not continuous at (0,0).
   (b) Find $f_x(0, 0)$ and $f_y(0, 0)$ (be careful to follow the definition of f closely).
   (c) Even though f_x and f_y exist at (0,0), why is f not differentiable at (0,0)?
   (d) Verify your results with a plot of the surface $z = f(x, y)$.

4. **Conservative Vector Fields.** Consider the vector field

$$\mathbf{F}(x, y) = \langle 4x^3y^2 - 2xy^3, 2x^4y - 3x^2y^2 + 4y^3\rangle.$$

   (a) Show that $\mathbf{F}$ is a conservative vector field
   (b) Find a function $f(x, y)$ so that $\nabla f = \mathbf{F}$.
   (c) Use the fact that $\mathbf{F}$ is conservative to evaluate the integral

$$\int_C \mathbf{F} \cdot d\mathbf{r}$$

   where the curve C is given by the vector function $\mathbf{r}(t) = \langle t + \sin \pi t, 2t + \cos \pi t\rangle$, $0 \leq t \leq 1$.

5. **Verifying Green's Theorem.** Let C be the triangle with vertices (0,0), (1,0) and (1,2). Evaluate the line integral $\displaystyle\int_C xy\, dx + x^2y^3\, dy$ in two ways: directly and with Green's Theorem.

6. **Green's Theorem in Theory.** Suppose f and g are continuously differentiable scalar-valued functions and C is a closed curve in the plane determined by the vector valued function $\mathbf{r}$. Use Green's Theorem and Clairaut's Theorem to show that

$$\int_C f(\mathbf{r})\nabla g(\mathbf{r}) \cdot d\mathbf{r} = -\int_C g(\mathbf{r})\nabla f(\mathbf{r}) \cdot d\mathbf{r}.$$

7. **Green's Theorem and Area.**

   (a) Use Green's Theorem to show that for any closed simple curve C oriented counterclockwise, the area inside C is equal to $\frac{1}{2}\int_C x\, dy - y\, dx$

   (b) Use part (a) to calculate the area inside the ellipse C which has parametric representation $\mathbf{r}(t) = \langle a\cos t, b\sin t\rangle,\ \ 0 \le t \le 2\pi$, where a and b are positive constants.

   (c) If C is the line segment connecting the point (x_1, y_1) to the point (x_2, y_2), show that

$$\int_C x\, dy - y\, dx = x_1y_2 - x_2y_1.$$

   (d) If the vertices of a polygon, in counterclockwise order, are (x_1, y_1), (x_2, y_2), $\ldots$, (x_n, y_n), show that the area of the polygon is

$$A = \frac{1}{2}\left[(x_1y_2 - x_2y_1) + (x_2y_3 - x_3y_2) + \cdots + (x_{n-1}y_n - x_ny_{n-1}) + (x_ny_1 - x_1y_n)\right].$$

   (e) Find the area of the pentagon with vertices $(0,0)$,$(2,1)$,$(1,3)$, $(0,2)$, and $(-1,1)$.

8. **Double Integrals and Joint Probability.** Let $f(x,y) = \frac{1}{4}e^{-(|x|+|y|)}$ be the joint probability density function for the population distribution of a city. The probability that a person selected at random lives in the region R is given by $\iint_R f(x,y)\, dA$, where x and y are measured in kilometers.

   (a) Find the probability that a person selected at random lives no more than 4 km north of Main Street ($y = 0$) and no more than 4 km east of Broadway (x=0).

   (b) Find the value of c such that the square bounded by the lines $x = y = -c$ and $x = y = c$ contains 50% of the population.

   (c) Explain, but do not calculate, how you would determine the probability that a randomly selected person lives on Main Street.

9. **Path Integration.**

(a) Suppose f is a continuous scalar function and let C be a level curve of f, i.e., $C = \{(x, y) \mid f(x, y) = k\}$ for some constant k. Express $\int_C f(x, y)\, ds$ in terms of k and the arclength of C.

(b) Let $f(x, y) = \begin{cases} 2 \text{ if } x + y \leq 1 \\ 1 \text{ otherwise} \end{cases}$

i. Let C_1 be the line segment running from (−2,0) to (2,0). Find $\int_{C_1} f(x, y)\, ds$

ii. Let C_2 be the upper half circle of radius 1 centered at (1,0). Find $\int_{C_2} f(x, y)\, ds$

10. **Integrals with Path Independence even for Functions with a Discontinuity.** Let $P(x, y) = \dfrac{-x}{(x^2 + y^2)^{3/2}}$ and $Q(x, y) = \dfrac{-y}{(x^2 + y^2)^{3/2}}$.

(a) Show that, except at the origin, $\dfrac{\partial P}{\partial y} = \dfrac{\partial Q}{\partial x}$.

(b) Find a potential function f so that $\nabla f = \langle P, Q \rangle$.

(c) Suppose C is a continuous curve running from (1,0) to (3,4) and not passing through the origin. Find $\int_C P\, dx + Q\, dy$

(d) Why were we able to use path independence here even though at $(0, 0)$ $\dfrac{\partial P}{\partial y} \neq \dfrac{\partial Q}{\partial x}$?

11. **An Integral Around Closed Curves for Functions with a Discontinuity.** Let $P(x, y) = \dfrac{-y}{x^2 + y^2}$ and $Q(x, y) = \dfrac{x}{x^2 + y^2}$.

(a) Show that, except at the origin, $\dfrac{\partial P}{\partial y} = \dfrac{\partial Q}{\partial x}$.

(b) Let C_1 be a circle of radius r centered at the origin with positive orientation. Find $\int_{C_1} P\, dx + Q\, dy$

(c) Let C_2 be a circle of radius r centered at a point so that the origin is not on or inside the circle. Find $\int_{C_2} P\, dx + Q\, dy$.

(d) Explain the results of parts (a) and (b).

12. **Polar Coordinates** A swimming pool is circular with diameter of 40 ft. The depth is constant in the east-west direction. The depth increases linearly from 2 ft at the south edge to 7 ft at the north edge. Use polar coordinates and a double integral to find the volume of the swimming pool.

13. **Change of variables** Consider the integral

$$\int\int x + 2y \, dA$$

on the triangular region with vertices $(3/2, 5/2), (4, 0), (-7/2, -5/2)$.

(a) Plot the triangular region and determine the equations of the lines that form the sides of the triangle. You'll notice that this region is neither of type I nor type II.

(b) Decompose the region into a collection of regions that are type I or type II and compute the integral. (I think you can find a decomposition with only two triangles.)

(c) Now consider the change of variables given by

$$x = u + v, \qquad y = u - v.$$

Find the resulting region in the variables (u, v) and compute the resulting integral, remembering to include the scaling factor given by the Jacobian determinant.