Calculus 3 Sample Exam 1, Spring 2018

1. **True or False, and Explain**

   (a) The vector $\langle 2, -1, 3\rangle$ is normal to the plane

   $$(x-2) - (y+1) + 3(z-3) = 0.$$

   (b) The lines

   $$\mathbf{r}_1(t) = \langle -5+t, 2-2t, -7+3t\rangle \text{ and } \mathbf{r}_2(t) = \langle 3-2t, 1-t, 2-t\rangle$$

   intersect.

   (c) The vectors $\langle 2, -1, 3\rangle$ and $\langle -1, 1, 1\rangle$ are orthogonal.

   (d) The vectors $\langle -1, 1, 1\rangle$ and $\langle 2, -2, 2\rangle$ are parallel.

2. A vector $\mathbf{v}$ of magnitude v makes an angle α with the positive x-axis, angle β with the positive y-axis, and angle γ with the positive z-axis. Show that

   $$\mathbf{v} = v\cos\alpha\,\mathbf{i} + v\cos\beta\,\mathbf{j} + v\cos\gamma\,\mathbf{k}$$

   The values $\cos\alpha$, $\cos\beta$, and $\cos\gamma$ are called the *direction cosines* of the vector $\mathbf{v}$. Show that

   $$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$$

3. Given two unit vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ with $\mathbf{v}_1 \cdot \mathbf{v}_2 = \frac{1}{2}$, define a sequence of vectors as

   $$\mathbf{v}_3 = \mathrm{proj}_{\mathbf{v}_1}\mathbf{v}_2, \quad \mathbf{v}_4 = \mathrm{proj}_{\mathbf{v}_2}\mathbf{v}_3, \quad \mathbf{v}_5 = \mathrm{proj}_{\mathbf{v}_3}\mathbf{v}_4, \ldots$$

   Draw a picture showing the sequence of vectors $\mathbf{v}_n$, write a general formula for $|\mathbf{v}_n|$ and find

   $$\sum_{n=1}^{\infty} |\mathbf{v}_n|.$$

4. **Planes** There are lots of possible exam questions that test your knowledge of lines and planes. Here are two, and please prepare as many others as you can find!

   (a) Show that the planes $x + y - z = 1$ and $2x - 3y + 4z = 5$ are neither parallel nor perpendicular. Find, correct to the nearest degree, the angle between these planes.

   (b) Find the distance between the planes $3x + y - 4z = 2$ and $3x + y - 4z = 24$.

5. Each edge of a cubical box has length 1 m. The box contains nine spherical balls with the same radius r. The center of one ball is at the center of the cube and it touches the other eight balls. Each of the other eight balls touches three sides of the box. Thus, the balls are tightly packed in the box. Find r.

6. Let $\mathcal{B}$ be a solid box with length L, width W, and height H. Let $\mathcal{S}$ be the set of all points that are a distance at most 1 from some point of $\mathcal{B}$. Express the volume of $\mathcal{S}$ in terms of L, W, and H.

7. Find an equation for the surface consisting of all points P for which the distance from P to the x-axis is twice the distance from P to the yz-plane. Identify the surface.

8. A circular helix can be parametrized by the function $\mathbf{r}(t) = \langle a\cos t, a\sin t, bt\rangle$ where a and b are positive constants.

   (a) Use this to show that the circular helix has constant curvature.

   (b) The DNA molecule has the shape of a double helix. The radius of each helix is about 10 angstroms (1 Å$= 10^{-8}$ cm). Each helix rises 34 Å during each complete turn, and there are 2.9×10^8 complete turns.

   (c) Find the length of each helix.

   (d) Compare this length to the simple approximation (number of turns)$\cdot$(circumference).

9. The position function of a spaceship is

$$\mathbf{r}(t) = (3+t)\mathbf{i} + (2+\ln t)\mathbf{j} + \left(7 - \frac{4}{t^2+1}\right)\mathbf{k}$$

and the coordinates of a space station are (6, 4, 9). The captain wants the spaceship to coast into the space station. When should the engines be turned off?

10. A disk of radius 1 is rotating in the counterclockwise direction at a constant angular speed ω. A particle starts at the center of the disk and moves toward the edge along a fixed radial segment so that its position at time $t \geq 0$ is given by $\mathbf{r}(t) = t\mathbf{R}(t)$, where

$$\mathbf{R}(t) = \cos\omega t\,\mathbf{i} + \sin\omega t\,\mathbf{j}$$

(a) Show that the velocity $\mathbf{v}$ of the particle is

$$\mathbf{v} = \cos\omega t\,\mathbf{i} + \sin\omega t\,\mathbf{j} + t\mathbf{v}_d$$

where $\mathbf{v}_d = \mathbf{R}'(t)$ is the velocity of a point on the edge of the disk.

(b) Show that the acceleration $\mathbf{a}$ of the particle is

$$\mathbf{a} = 2\mathbf{v}_d + t\mathbf{a}_d$$

where $\mathbf{a}_d = \mathbf{R}''(t)$ is the acceleration of a point on the rim of the disk. The extra term $2\mathbf{v}_d$ is called the *Coriolis acceleration*; it is the result of the interaction of the rotation of the disk and the motion of the particle. One can obtain a physical demonstration of this acceleration by walking toward the edge of a moving merry-go-round. Draw a diagram showing the motion of the particle and the components of the acceleration.

11. A team of engineers has been hired to redesign a curve in a road so that the curve can be traveled safely at a higher speed. The current radius of the curve is 40 yards, and has been designed to be safe at a speed limit of 30 mph. The state has raised the speed limit on this road from 30 to 45 mph. How large must the new radius be in order for the curve to be safe at 45 mph? Explain your reasoning.