Calculus III Sample Exam 2, Spring 2018

1. + If $u = x^y$ show that $\dfrac{x}{y}\dfrac{\partial u}{\partial x} + \dfrac{1}{\ln x}\dfrac{\partial u}{\partial y} = 2u$

   HINT: remember that $d/dx(a^x) = a^x \ln a$.

2. The equations $z = f(x, y)$ and $F(x, y, z) = f(x, y) - z = 0$ describe the same surface in space. Yet ∇f and ∇F are not the same vectors.

   (a) What is the difference? In what way are the two gradients related?

   (b) Give a formula for the tangent plane to f at a point $(x_0, y_0, z_0 = f(x_0, y_0))$.

   (c) Give a formulat for the tangent plane to the normal surface $F = 0$ at $(x_0, y_0, z_0 = f(x_0, y_0))$. Explain the connection with your answer in part 2b.

   (d) + Show that the ellipsoid $3x^2 + 2y^2 + z^2 = 9$ and the sphere $x^2 + y^2 + z^2 - 8x - 6y - 8z + 24 = 0$ are tangent to each other at the point $(1, 1, 2)$, that is, they have a common tangent plane at the point.

3. At a given point, the directional derivatives of $f(x, y)$ are known in two non-parallel directions, given by unit vectors $\mathbf{u}$ and $\mathbf{v}$. Can you determine ∇f at this point? If so, how do you do it?

4. You are given only the following information about a function f:

   $$f(8, 5) = 33.1 \qquad f(8.01, 5) = 33.3 \qquad f(8, 5.02) = 33.0.$$

   (a) Approximate the equation of the tangent plane to the surface $z = f(x, y)$ at (8,5).

   (b) Given only the three pieces of information about f in this problem, it is impossible to find the value of f at the point (8.01,5.02); however, you can *approximate* the value of f there. Do so.

5. + Let $T(x, y) = x^2 - 2xy$ be the temperature at the point (x, y) in the region bounded by the curves $y = x$ and $y = x^2$. Suppose that a bug is crawling around inside the region.

   (a) At $(1/2, 1/3)$, in what direction should the bug go to cool down as quickly as possible?

   (b) At $(1/2, 1/3)$, in what directions should the bug go to maintain its current temperature?

   (c) What is the hottest point in the region?

   (d) If, at $(1/2, 1/3)$, the bug begins to move directly toward the origin, does it experience a rise, a fall, or no change in temperature?

6. (a) + Find the absolute maximum and minimum values of the function $f(x, y) = 1 + 4x - 5y$ on the closed rectangular region with vertices $(0, 0)$, $(2, 0)$, $(2, 3)$, and $(0, 3)$.

   (b) Explain why if $\nabla f \neq 0$ for all points in a closed region, the absolute maximum and minimum values of f on the region must occur on the boundary.

7. The plane $4x - 3y + 8z = 5$ intersects the cone $z^2 = x^2 + y^2$ in an ellipse.

   (a) Graph the cone and the plane together in one plot

   (b) Find the parametric equations for the tangent line to this ellipse at the point $(0, 1, 1)$. HINT: the tangent line will be on the tangent planes to both surfaces at this point.

   (c) Find the highest and lowest points on the ellipse. HINT: set this up as an optimization problem for the (very simple) height function with two constraints, and then solve the resulting system of equations with Maxima

8. (a) Find the positive values x, y and z that maximize the function

$$f(x, y, z) = \sqrt[3]{xyz}$$

subject to the constraint

$$x + y + z = C$$

where C is a constant.

(b) Use part (a.) to deduce that

$$\sqrt[3]{xyz} \leq \frac{x + y + z}{3}.$$

(c) Explain how you could redo (a.) and (b.) to show that for any n

$$\sqrt[n]{x_1 x_2 \cdots x_n} \leq \frac{x_1 + x_2 + \cdots + x_n}{n}.$$

In this way, you have proved that the geometric mean is always less than or equal to the algebraic mean.

9. + Consider a plane that cuts off a portion of the 1st octant, intersecting the coordinate axes at $(a, 0, 0)$, $(0, b, 0)$, and $(0, 0, c)$. Use a double integral to show that the volume in the first octant below the plane is $abc/6$, and confirm that this is consistent with the well-known formula for the volume of a pyramid as one-third area of the base times the height.

10. + Suppose two random variables X and Y can take values $0 \leq X \leq 3$ and $0 \leq Y \leq 2$. Their joint density function is

$$f(x, y) = \begin{cases} C(x + y) & \text{if } 0 \leq x \leq 3, \;\; 0 \leq y \leq 2 \\ 0 & \text{otherwise} \end{cases}$$

(a) An important property of a joint density function is that when integrated over all possible values of X and Y, the result is 1. In this case we denote that as

$$1 = P(0 \leq X \leq 3, 0 \leq Y \leq 2) = \int_{y=0}^{y=2} \int_{x=0}^{x=3} f(x, y)\, dx\, dy.$$

Use that fact about $f(x, y)$ to find the value of the constant C.

(b) Find $P(X \leq 2, Y \geq 1)$

(c) Find $P(X + Y \leq 1)$