New types of 3-D systems of quadratic differential equations with chaotic dynamics based on Ricker discrete population model

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A B S T R A C T

The wide class of 3-D autonomous systems of quadratic differential equations, in each of which either there is a couple of coexisting limit cycles or there is a couple of coexisting chaotic attractors, is found. In the second case the couple consists of either Lorenz-type attractor and another attractor of a new type or two Lorenz-type attractors. It is shown that the chaotic behavior of any system of the indicated class can be described by the Ricker discrete population model: \( z_{i+1} = z_i \exp(r - z_i), \ r > 0, \ z_i > 0, \ i = 0, 1, \ldots \). The values of parameters, at which in the 3-D system appears either the couple of limit cycles or the couple of chaotic attractors, or only one limit cycle, or only one sphere-shaped chaotic attractor, are indicated. Examples are given.

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1. Introduction

In 1963 Lorenz discovered a simple 3-D autonomous chaotic system. This system has only two quadratic non-linearity terms. In future many of other chaotic systems have been found. One of principal directions of searches was the finding it is possible more simple chaotic systems with quadratic non-linearities.

It should be noted that most found chaotic systems had the kind:

\[
\begin{align*}
\dot{x}(t) &= a_{11}x(t) + a_{12}y(t) + a_{13}z(t) + b_{11}y(t)z(t) + b_{12}x(t)z(t) + b_{13}x(t)y(t), \\
\dot{y}(t) &= a_{21}x(t) + a_{22}y(t) + a_{23}z(t) + b_{21}y(t)z(t) + b_{22}x(t)z(t) + b_{23}x(t)y(t), \\
\dot{z}(t) &= a_{31}x(t) + a_{32}y(t) + a_{33}z(t) + b_{31}y(t)z(t) + b_{32}x(t)z(t) + b_{33}x(t)y(t),
\end{align*}
\]

(1)

where at least one of coefficients \( b_{ij} \neq 0; \ i, j \in \{1, 2, 3\} \).

Systems (1) are investigated in [1–14]. It is possible to one of the first works, in which a term \( x^2 \) appeared, there was article [15]. In all indicated works for systems of form (1) problems typical for the nonlinear analysis are solved: positions of equilibrium were investigated, Lyapunov exponents were calculated, bifurcation diagrams were built. In addition the next important questions were raised. How many chaotic attractors are contained in system (1)? How many scrolls and wings are there in given attractor?

It is known that both Lorenz system and Chen system possess a two-wing attractor. So many wings have an attractor given in [2]. A single three-wing or four-wing chaotic attractor was generated in papers [10,12]. Attractors with 2, 3 and 4 scrolls were found in [1,4].

There are also \( k \)-dimensional chaotic systems \( k > 3 \), which have more complicated dynamic than 3-D chaotic systems [15–19]. Besides, there is a large class of chaotic systems are based on the theory of homoclinic and heteroclinic orbits [20–25]. 3-D quadratic systems with any non-linearities can be included in this class. However it should be said that possibly only in paper [12] the system containing two coexisting chaotic single-wing attractors was indicated.

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It is known that chaos is very useful in many application fields: engineering, medicine, secure communications, and so on. It is very exciting to construct a lower-dimensional chaotic system which has a simple algebraic structure, but with a complex attractor structure (the number of scrolls or wings of the attractor are more than two).

In this connection we will consider the following system:

\[
\begin{aligned}
\dot{x}(t) &= a_{11}x(t) + a_{12}y(t) + a_{13}z(t) + h_{11}y^2(t) + h_{12}y(t)z(t) + h_{22}z^2(t), \\
\dot{y}(t) &= a_{21}x(t) + a_{22}y(t) + a_{23}z(t) + x(t)y(t) + b_2z(t), \\
\dot{z}(t) &= a_{31}x(t) + a_{32}y(t) + a_{33}z(t) + x(t)c_1y(t) + c_2z(t).
\end{aligned}
\]  

(2)

In the present work systems (2) containing couple of coexisting chaotic two-wing attractors will be found (one of this couple is the Lorenz-type attractor). It will be shown that the chaotic behavior of any system of the indicated class can be described by the Ricker discrete population model \([26–28]\). Besides, systems (2) containing couple of coexisting limit cycles, or one Lorenz-type attractor and one limit cycle, either only one limit cycle or only one sphere-shaped chaotic attractor, will be also indicated.

2. Bounded solutions of homogeneous quadratic systems

We will consider the following homogeneous 3-D system

\[
\begin{aligned}
\dot{x}(t) &= h_{11}y^2(t) + h_{12}y(t)z(t) + h_{22}z^2(t), \\
\dot{y}(t) &= x(t)(b_1y(t) + b_2z(t)), \\
\dot{z}(t) &= x(t)c_1y(t) + c_2z(t)
\end{aligned}
\]  

(3)

with initial values \(x(0) = x_0, y(0) = y_0, z(0) = z_0\). (Here \(h_{11}, h_{12}, h_{22}, b_1, b_2, c_1, c_2 \in \mathbb{R}\)).

Introduce the matrix

\[
D = \begin{pmatrix} b_1 & b_2 \\ c_1 & c_2 \end{pmatrix}
\]

Let \(\det D \neq 0\). Then from the practical point of view only two following cases are interesting: the eigenvalues of the matrix \(D\) are real and distinct; the eigenvalues of the matrix \(D\) are complex.

By means of approaching a linear invertible real replacement of variables \(y\) and \(z\) the matrix \(D\) can be reduced either to the form

\[
D = \begin{pmatrix} b & 0 \\ 0 & c \end{pmatrix}
\]

or the form

\[
D = \begin{pmatrix} p & q \\ -q & p \end{pmatrix}
\]

Thus, system (3) may be transformed either to the system

\[
\begin{aligned}
\dot{x}(t) &= a_{11}x^2(t) + a_{12}y(t)z(t) + a_{22}z^2(t), \\
\dot{y}(t) &= bx(t)y(t), \\
\dot{z}(t) &= cx(t)z(t)
\end{aligned}
\]  

(4)

or the system

\[
\begin{aligned}
\dot{x}(t) &= a_{11}y^2(t) + a_{12}y(t)z(t) + a_{22}z^2(t), \\
\dot{y}(t) &= x(t)py(t) + qz(t), \\
\dot{z}(t) &= x(t)(-qy(t) + pz(t)).
\end{aligned}
\]  

(5)

(For simplicity we have left the former designations of variables \(x, y\) and \(z\); \(a_{11}, a_{12}, a_{22} \in \mathbb{R}\).)

**Theorem 1.** Let \(a_{11}a_{22}bc \neq 0\). Assume that for system (4) the following conditions

\[
\frac{a_{11}}{b} < 0, \quad \frac{a_{22}}{c} < 0
\]

are satisfied. If either

\[
b + c \neq 0, \quad \frac{a_{12}^2}{(b + c)^2} - \frac{a_{11}a_{22}}{bc} < 0
\]
or
\[ b + c = 0, \quad a_{12} = 0, \]
then all solutions of system (4) are bounded at any initial values.

**Proof.** From the second and third equations of system (4) it follows that \( y(t) = c_0 z^k(t) \), where \( k = b/c \) and \( c_0 = y_0/z^k \). Then system (4) may be reduced to the form
\[
\begin{align*}
\dot{x}(t) &= a_{11} c_0 z^{2k}(t) + a_{12} c_0 z^{k+1}(t) + a_{22} z^2(t), \\
\dot{z}(t) &= c\dot{x}(t) z(t).
\end{align*}
\] (6)

An elementary integration of system (6) reduces to the relationship
\[
\frac{c}{2} x^2(t) - a_{11} c_0^2 z^{2k}(t) - a_{12} c_0^2 z^{k+1}(t) - a_{22} z^2(t) = c_1(x_0, y_0, z_0) = \text{const}. \tag{7}
\]

Let \( v = z^{k-1} \). Then from (7) it follows that
\[
x^2(t) + z^2(t) \cdot \left( a_{11} c_0^2 \frac{v^2(t)}{c} - 2 a_{12} c_0 \frac{v(t)}{c(k+1)} - a_{22} \frac{1}{c} \right) = c_2(x_0, y_0, z_0) = \text{const}.
\]

Let \( k = -1 \). Then for boundedness of solutions of system (4) it is necessary and sufficiently that
\[
\frac{a_{11}}{ck} < 0, \quad \frac{a_{22}}{c} < 0, \quad \frac{a_{12}^2}{(k+1)^2} - \frac{a_{11} a_{22}}{k} < 0.
\]
(The quadratic function with respect to the variable \( v \) in the big round brackets is positive.)

If \( k = -1 \), then for boundedness of solutions of system (4) it is necessary and sufficiently that conditions
\[
a_{12} = 0, \quad \frac{a_{11}}{ck} < 0, \quad \frac{a_{22}}{c} < 0
\]
was fulfilled. \( \square \)

**Theorem 2.** Let \((a_{11} - a_{22})^2 + a^2_{12} \neq 0 \) and \( p q \neq 0 \). If for system (5)
\[
\frac{\sqrt{p^2 + q^2}}{p} \frac{a_{11} + a_{22}}{\sqrt{(a_{11} - a_{22})^2 + a^2_{12}}} < -1,
\]
then all solutions of this system is bounded at any initial values. Besides, for some initial values system (5) has periodic solutions.

**Proof.** Introduce into system (5) new variables \( \rho \) and \( \phi \) under the formulas: \( y = \rho \cos \phi, \quad z = \rho \sin \phi \), where \( \rho > 0 \). Then we get
\[
\begin{align*}
\dot{x}(t) &= \rho^2(t) \cdot \left( a_{11} \cos^2 \phi(t) + a_{12} \cos \phi(t) \sin \phi(t) + a_{22} \sin^2 \phi(t) \right), \\
\dot{\rho}(t) &= \rho \cdot \dot{x}(t) \cdot \rho(t), \\
\dot{\phi}(t) &= -q \cdot \dot{x}(t).
\end{align*}
\] (8)

From the second and third equations of system (8) it follows that \( \rho(\phi) = \rho_0 \exp \left( -\frac{2p}{q} \phi \right) \), where \( \rho_0 = \rho(\phi_0) = \text{const} > 0 \). In this case system (8) may be reduced to the form
\[
\begin{align*}
\dot{x}(t) &= \rho^2_0 \exp \left( -\frac{2p}{q} \phi(t) \right) \cdot \left( a_{11} \cos^2 \phi(t) + a_{12} \cos \phi(t) \sin \phi(t) + a_{22} \sin^2 \phi(t) \right), \\
\dot{\phi}(t) &= -q \cdot \dot{x}(t).
\end{align*}
\] (9)

An integration of system (9) reduces to the relationships
\[
\begin{align*}
\frac{q x^2}{2} + \rho^2_0 a_{11} &\int \exp \left( -\frac{2p}{q} \phi \right) \cos^2 \phi \, d\phi + \rho^2_0 a_{12} \int \exp \left( -\frac{2p}{q} \phi \right) \cos \phi \sin \phi \, d\phi + \rho^2_0 a_{22} \int \exp \left( -\frac{2p}{q} \phi \right) \sin^2 \phi \, d\phi \\
&= c_1(x_0, \rho_0, \phi_0) = \text{const},
\end{align*}
\]
\[
\begin{align*}
\frac{q x^2}{2} - \frac{q^2}{4p} (a_{11} + a_{22}) &\exp \left( -\frac{2p}{q} \phi \right) + \frac{q^2}{4(p^2 + q^2)} (a_{11} - a_{22}) \left( \sin 2\phi - \frac{p}{q} \cos 2\phi \right) \exp \left( -\frac{2p}{q} \phi \right) \\
&- \frac{q^2}{4(p^2 + q^2)} (\cos 2\phi + \frac{p}{q} \sin 2\phi) \exp \left( -\frac{2p}{q} \phi \right) = c_1(x_0, \rho_0, \phi_0) = \text{const}.
\end{align*}
\] (10)
Let $q > 0$. As $\exp(-2p\phi/q) > 0$, then the left part of relationship (10) will be positive if

$$-rac{p^2 + q^2}{p} (a_{11} + a_{22}) + q \left( (a_{11} - a_{22}) - a_{12} \frac{p}{q} \right) \sin 2\phi - q \left( a_{11} - a_{22} \right) \frac{p}{q} + a_{12} \cos 2\phi > 0. \quad (11)$$

Introduce the angle $\theta$ under the formulas

$$\cos \theta = \frac{(a_{11} - a_{22})p + a_{12}q}{\sqrt{[(a_{11} - a_{22})p + a_{12}q]^2 + [(a_{11} - a_{22})q - a_{12}p]^2}},$$

$$\sin \theta = \frac{(a_{11} - a_{22})q - a_{12}p}{\sqrt{[(a_{11} - a_{22})p + a_{12}q]^2 + [(a_{11} - a_{22})q - a_{12}p]^2}}.$$

Then inequality (11) may be rewritten in the form

$$-rac{p^2 + q^2}{p} (a_{11} + a_{22}) - \sqrt{[(a_{11} - a_{22})p + a_{12}q]^2 + [(a_{11} - a_{22})q - a_{12}p]^2} \cos(2\phi + \theta) > 0. \quad (12)$$

As $|\cos(2\phi + \theta)| \leq 1$, then from (12) it follows that

$$\frac{\sqrt{p^2 + q^2}}{p} \frac{a_{11} + a_{22}}{\sqrt{(a_{11} - a_{22})^2 + a_{12}^2}} < -1.$$

Suppose that $\forall \phi$ inequality (11) is fulfilled. Then formula (10) may be rewritten as

$$q \frac{x^2}{2} + q \rho^2 \cdot f(\phi) = c_1(x_0, \rho_0, \phi_0) = \text{const} > 0,$$

where $\forall \phi$ the function

$$f(\phi) = -\frac{a_{11} + a_{22}}{4p} + \frac{q(a_{11} - a_{22})}{4(p^2 + q^2)} \left( \sin 2\phi - \frac{p}{q} \cos 2\phi \right) - \frac{qa_{12}}{4(p^2 + q^2)} \left( \cos 2\phi + \frac{p}{q} \sin 2\phi \right)$$

is positive.

Similar result may be derived if $q < 0$. From here it follows that the functions $x(\phi)$ and $\rho(\phi)$ are bounded. From (9) it follows that the function $\phi(t)$ changes a sign if $x(t) = 0$. Hence, we have $\dot{\phi}(t) = 0$ at $x(t) = 0$. It is means that if $f(\phi) > 0$, then at $x(t) = 0$ extremums of the function $\phi(t)$ take a finite sequence

$$\phi_0 < \phi_1 < \cdots < \phi_{m-1} < \phi_m$$

values each of which are defined with the equation

$$f(\phi) = \frac{c_1}{q} \exp \left( \frac{2p}{q} \phi \right). \quad (13)$$

(Since the function $f(\phi)$ is periodic and nonnegative, then for $c_1/q > 0$ Eq. (13) has always a solution.)

It is possible to conclude from the analysis of Eq. (13), that for system (8) there are so-called periodicity intervals. (The periodicity interval is a segment $I_k = [\phi_{0k}, \phi_{k+1}]$ of a numerical axis such that at $x = 0$, $\forall \rho > 0$ and $\forall \phi \in I_k$ system (8) has a periodic solution.) These intervals are shown on Fig. 1.

Let $\phi_0 \in I_k$ (from Fig. 1 we can takes $\phi_0 = 1.56$). Then $\rho_0^2 = y_0^2 + z_0^2 = 2$, $\tan \phi_0 = \tan 1.56 = z_0/y_0 = 1.0$. From here it follows that $y_0 = z_0 = 1$.

Thus, for some initial values system (5) has periodic solutions. \hfill \Box

3. Limit cycles and torus of nonhomogeneous quadratic systems

Consider the following system

$$\begin{aligned}
\dot{x}(t) &= ax(t) + a_{11}y^2(t) + a_{12}yz(t) + a_{22}z^2(t), \\
\dot{y}(t) &= by(t) + cz(t) + x(t)(py(t) + qz(t)), \\
\dot{z}(t) &= -cy(t) + bz(t) + x(t)(-qy(t) + pz(t)).
\end{aligned} \quad (14)
$$
Proof. Introduce into system (14) new variables \(\rho\) and \(\phi\) under the formulas: \(y = \rho \cos \phi, z = \rho \sin \phi\), where \(\rho > 0\). Then we get

\[
\begin{align*}
\dot{x}(t) &= a \cdot x(t) + \rho^2(t) \cdot \left(a_{11} \cos^2 \phi(t) + a_{12} \cos \phi(t) \sin \phi(t) + a_{22} \sin^2 \phi(t)\right), \\
\dot{\rho}(t) &= (b + p \cdot x(t)) \cdot \rho(t), \\
\dot{\phi}(t) &= -(c + q \cdot x(t)).
\end{align*}
\]

From the second and third equations of system (15) it follows that

\[\rho(\phi, t) = \rho_0 \exp \left(-\frac{p}{q} \phi + \left(b - \frac{p}{q} c\right) t\right),\]

where \(\rho_0 = \rho(\phi_0) = \text{const} > 0\). In this case system (15) may be reduced to the form

\[
\begin{align*}
\dot{x}(t) &= a \cdot x(t) + \rho_0^2 \exp \left(-\frac{2p}{q} \phi(t)\right) \cdot \exp \left(2\left(b - \frac{p}{q} c\right)t\right) \cdot (a_{11} \cos^2 \phi(t) \\
&\quad + a_{12} \cos \phi(t) \sin \phi(t) + a_{22} \sin^2 \phi(t)) \equiv f(x, \phi, t), \\
\dot{\phi}(t) &= -(c + q \cdot x(t)) \equiv g(x).
\end{align*}
\]

Theorem 3. Let for system (14) \(a < 0, b > 0\), and \(b - (p/q)c < 0\). Then under the conditions of Theorem 2 in system (14) there exists either a limit cycle or a limit torus.

(a) Let \(a = 0\). Then we can get the solution \(x(t)\) of system (16) in the form

\[
x(t) = x_0 + \rho_0^2 \int_0^t \exp \left(-\frac{2p}{q} \phi(\tau)\right) \cdot \exp \left(2\left(b - \frac{p}{q} c\right)\tau\right) \times \left(a_{11} \cos^2 \phi(\tau) + a_{12} \cos \phi(\tau) \sin \phi(\tau) + a_{22} \sin^2 \phi(\tau)\right) d\tau.
\]

As \(0 < \exp(2(b - (p/q)c)t) < 1\), then from (17) it follows that

\[
|x(t)| \leq |x_0| + \rho_0^2 \int_0^t \exp \left(-\frac{2p}{q} \phi(\tau)\right) \left|a_{11} \cos^2 \phi(\tau) + a_{12} \cos \phi(\tau) \sin \phi(\tau) + a_{22} \sin^2 \phi(\tau)\right| d\tau.
\]

Assume that \(b - (p/q)c = 0\). Then system (16) may be written in the following aspect:

\[
\begin{align*}
\dot{z}(t) &= \rho_0^2 \exp \left(-\frac{2p}{q} \phi(t)\right) \cdot (a_{11} \cos^2 \phi(t) + a_{12} \cos \phi(t) \sin \phi(t) + a_{22} \sin^2 \phi(t)), \\
\dot{\phi}(t) &= -(c + q \cdot z(t)).
\end{align*}
\]
After integration of system (19), we will have
\[
\frac{q^2}{2} z_2 + qz - \frac{p^2}{4p}(a_{11} + a_{22}) \exp \left( - \frac{2p}{q} \phi \right) + \frac{p^2q^2}{4(p^2 + q^2)} \left( \sin 2\phi - \frac{p}{q} \cos 2\phi \right) \exp \left( - \frac{2p}{q} \phi \right) 
- \frac{\rho^2q^2}{4(p^2 + q^2)} \left( \cos 2\phi + \frac{p}{q} \sin 2\phi \right) \exp \left( - \frac{2p}{q} \phi \right) = \text{const.}
\]

Thus, the boundedness conditions of solutions of Eq. (19) will be the same as well as for Eq. (9).

For solutions of system (19), we obtain the following estimate:
\[
|z(t)| \leq |z_0| + \rho_0 t \int_0^t \exp \left( - \frac{2p}{q} \phi(\tau) \right) a_{11} \cos^2 \phi(\tau) + a_{12} \cos \phi(\tau) \sin \phi(\tau) + a_{22} \sin^2 \phi(\tau) \, d\tau.
\]

This inequality is equivalent to (18). Therefore, from boundedness of the solution \( z(t) \) it follows the same property of the solution \( x(t) \) of system (16) at \( a \leq 0 \).

(b) Let \( a < 0 \). From the first equation of system (15), we have
\[
x(t) = x_0 \exp(\alpha t) + \int_0^t \exp(\alpha(t-\tau)) \rho^2(\tau)(a_{11} \cos^2 \phi(\tau) + a_{12} \cos \phi(\tau) \sin \phi(\tau) + a_{22} \sin^2 \phi(\tau)) \, d\tau.
\]

From here it follows that
\[
|x(t)| < |x_0| + \int_0^t \rho^2(\tau) a_{11} \cos^2 \phi(\tau) + a_{12} \cos \phi(\tau) \sin \phi(\tau) + a_{22} \sin^2 \phi(\tau) \, d\tau.
\]

For \( b - (p/q)c < 0 \) the last inequality is equivalent to (18). Thus, the solutions of system (15) are bounded.

(c) Let’s calculate Lyapunov’s exponent \( \Lambda \) for a real function \( f(t) \):
\[
\Lambda[f] = \lim_{t \to \infty} \frac{1}{t} \ln \left| \frac{f(t)}{f(0)} \right|
\]

We take advantage of the following properties Lyapunov’s exponents:

1. If \( f(t+\phi) = f(t) + \phi \), then \( \Lambda[f(\phi)] = \Lambda[f] \);
2. If \( \phi(t) \) is a constant function, then \( \Lambda[f(\phi)] = 0 \);
3. If \( \phi(t) \) is a periodic function with period \( T \), then \( \Lambda[f(\phi)] = \frac{1}{T} \int_0^T \Lambda[f(\phi)] \, d\tau \);
4. If \( f(t) \) is a periodic function with period \( T \), then \( \Lambda[f(\phi)] = \max(\Lambda[f(T)], \Lambda[f(\phi)]) \);
5. If \( \phi(t) \) is a periodic function with period \( T \), then \( \Lambda[f(\phi)] = \max(\Lambda[f(T)], \Lambda[f(\phi)]) \).

First we calculate \( \Lambda[\rho] \). It is clear that \( \max_{t \to \infty} \rho(t) \) may be reached under the condition \( \dot{\rho}(t) = 0 \) or at \( x = -b/p \) (it follows from the second equation of system (15)). In this case from the third equation of system (16) it follows that \( \lim_{t \to \infty} \rho(t) \to \max \) if \( \lim_{t \to \infty} \phi(t) = -c + b(q/p) \). Thus,
\[
\Lambda[\rho] = \lim_{t \to \infty} \frac{1}{t} \ln \left| \frac{\rho(t)}{\rho_0} \right| = \lim_{t \to \infty} \frac{1}{t} \ln \left| \frac{\rho_0 \exp \left( -\frac{2p}{q} \phi(t) + \frac{b}{q} \right)}{\rho_0} \right| = -\frac{p}{q} \lim_{t \to \infty} \frac{\phi(t)}{t} + b - \frac{p}{q} c.
\]

Using L’Hospital’s rule, we get
\[
\Lambda[\rho] = -\frac{p}{q} \lim_{t \to \infty} \frac{\phi(t)}{t} + b - \frac{p}{q} c = -\frac{p}{q} \lim_{t \to \infty} \phi(t) + b - \frac{p}{q} c = 0.
\]

Consider the function
\[
h(\phi) = a_{11} \cos^2 \phi + a_{12} \cos \phi \sin \phi + a_{22} \sin^2 \phi + \frac{a_{11} + a_{22}}{2} \cos 2\phi + \frac{a_{11} - a_{22}}{2} \sin 2\phi
\]
\[
= \frac{a_{11} + a_{22}}{2} \sqrt{(a_{11} - a_{22})^2 + a_{22}^2} \sin(2\phi + \omega),
\]

where
\[
\sin \omega = \frac{a_{11} - a_{22}}{(a_{11} - a_{22})^2 + a_{22}^2}, \quad \cos \omega = \frac{a_{12}}{\sqrt{(a_{11} - a_{22})^2 + a_{22}^2}}.
\]
It is clear that \( h(\phi) \neq 0 \), if
\[
\frac{|a_{11} + a_{22}|}{\sqrt{(a_{11} - a_{22})^2 + a_{12}^2}} > 1.
\]

In this case Lyapunov’s exponent \( \Lambda(h) = 0 \). If
\[
\frac{|a_{11} + a_{22}|}{\sqrt{(a_{11} - a_{22})^2 + a_{12}^2}} \leq 1,
\]
then there exists a point \( \gamma \) such that \( h(\gamma) = 0 \). In this case \( \Lambda(h) = -\infty \).
From here and (c1) it follows that \( \Lambda(\exp(at^2h) \leq \Lambda(\exp(at)) + \Lambda(\rho^2) + \Lambda(h) \leq a + 0 + 0 = a \) or \( \Lambda(\exp(at^2h) = -\infty \).

From the third equation of system (15), we get
\[
K \exp \left( \int_t^x \exp(a(t - \tau))\rho^2(\tau)h(\tau)d\tau \right) \leq a.
\]

Therefore, from (20) and (c4), we have \( \Lambda[x(t)] \leq a \).
From the third equation of system (15), we get \( \phi(t) = \phi_0 - ct + q \int_0^t x(\tau)d\tau \). Then from (c4)–(c5) it follows that
\[
\Lambda[\phi(t)] \leq \Lambda[\phi_0 - ct] + \Lambda \left[ q \int_0^x x(t) \right] \leq \max(0,a) = 0.
\]

As \( a < 0 \), then \( \Lambda[x] + \Lambda[\rho] + \Lambda[\phi] \leq a < 0 \). It is means that system (15) or (14) is dissipative.

The origin is unique equilibrium of system (14). If \( b < 0 \) then the origin is a stable node. Therefore, by virtue of boundedness of solutions it will be attracted to the origin. If \( b > 0 \), then the origin is a saddle-focus, and from conditions \( \Lambda[x] \leq a, \Lambda[\rho] = 0, \Lambda[\phi] \leq 0 \) it follows that in system (14) there is either a limit cycle or a limit torus.

Different limit cycles and torus are represented on Figs. 2–5.
For greater visualization equilibriums of the following two systems displaced with respect to origin of coordinates.
Consider the system
\[
\begin{align*}
\dot{x}(t) &= 1000 - 3x(t) - 1000y^2(t) + 10z^2(t),
\dot{y}(t) &= y(t) + 2z(t) + x(t)y(t) + (4/3)z(t),
\dot{z}(t) &= -2y(t) + z(t) + x(t)(-4/3)y(t) + z(t).
\end{align*}
\]

Its behavior is shown on Fig. 4.
Consider such system
\[
\begin{align*}
\dot{x}(t) &= 2000 - 3x(t) - 300y^2(t) - 10000z^2(t),
\dot{y}(t) &= y(t) + 2z(t) + x(t)y(t) + (4/3)z(t),
\dot{z}(t) &= -2y(t) + z(t) + x(t)(-4/3)y(t) + z(t).
\end{align*}
\]

Its behavior is shown on Fig. 5.

**Fig. 2.** The limit cycle of system (8) for \( a = -0.3, a_{11} = -3, a_{12} = 0, a_{22} = 1, b = 2, c = 5, p = 1, q = 2, \) and \( t = 150 \).
4. Chaotic attractors of quadratic systems: case $b + c = 0$

Consider the following system

\[
\begin{align*}
\dot{x}(t) &= a_1 x(t) + a_{11} y(t)^2 + a_{12} y(t) z(t) + a_{22} z(t)^2, \\
\dot{y}(t) &= b_1 y(t) + c_1 z(t) + b x(t) y(t), \\
\dot{z}(t) &= -c_1 y(t) + b_1 z(t) + c x(t) z(t).
\end{align*}
\]

Theorem 4. Let for system (23) $a_i > 0$, $b_i < 0$, and $a_i + 2b_i < 0$. Then under the conditions of Theorem 1 and at $b + c = 0$ any solution of system (23) is dissipative. Besides there exists a set $W := \{x_0, y_0, z_0\}$ of initial values such that $\forall (x_0, y_0, z_0) \in W$ the solution of system (23) with initial values $(x_0, y_0, z_0)$ is bounded.

Proof

(a) We rewrite system (23) in the form:

\[\dot{x} = F(x), \quad x = (x, y, z)^T, \quad F : \mathbb{R}^3 \to \mathbb{R}^3.\]
Then a divergence of the field \( \mathbf{F}(x) \) can be calculated on the formula: \( \text{div} \mathbf{F}(x) = a_1 + 2b_1 < 0 \). It means that system (23) is dissipative.

(b) Assume \( b_1 = c_1 = 0 \). Without loss of generality it is possible to consider that \( a_{11} < 0, a_{12} = 0, a_{22} > 0 \). In this case it must be \( b > 0 \) and \( c < 0 \). Taking advantage of the change of variable \( x \to x/(2b) \), we can consider that \( b = 1, c = -1 \). Under these conditions from (23) it follows that

\[
\dot{x}(t) = a_1 x(t) + a_{11} y_0^2 \exp \left( \int_{t_0}^t x(\tau) d\tau \right) + a_{22} z_0^2 \exp \left( -\int_{t_0}^t x(\tau) d\tau \right).
\]

By definition, put \( V(t) = \dot{x}(t) - a_1 x(t) \), where the function \( V(t) \) is defined on an interval \([t_0, t] \in [0, \infty)\). Then we get

\[
V(t) = a_{11} y_0^2 \exp \left( \int_{t_0}^t x(\tau) d\tau \right) + a_{22} z_0^2 \exp \left( -\int_{t_0}^t x(\tau) d\tau \right).
\] (24)

Compute a derivative with respect to \( t \) from the function \( V(t) \). With the help of Eq. (24), we exclude the expressions \( \exp \left( \pm \int_{t_0}^t x(\tau) d\tau \right) \) from the formula

\[
\dot{V}(t) = a_{11} y_0^2 x(t) \exp \left( \int_{t_0}^t x(\tau) d\tau \right) - a_{22} z_0^2 x(t) \exp \left( -\int_{t_0}^t x(\tau) d\tau \right).
\]

Then we have \((\dot{V}(t))^2 = x^2(t) \left( V^2(t) - 4a_{11} a_{22} y_0^2 z_0^2 \right)\). From here it follows that

\[
\dot{V}(t) = \pm x(t) \sqrt{V^2(t) + g}, \quad g = \frac{-4a_{11} a_{22} y_0^2 z_0^2}{a_1^2} > 0.
\]

The last expression may be transformed to

\[
\pm \int_{t_0}^t x(\tau) d\tau = \int_{t_0}^t \frac{dV}{\sqrt{V^2 + g}} = \ln \frac{V(t) + \sqrt{V^2(t) + g}}{V(t_0) + \sqrt{V^2(t_0) + g}}.
\] (25)

By substituting (25) into (24) \( \forall t \in [t_0, \infty) \), we obtain

\[
V(t) = a_{11} y_0^2 \frac{V(t) + \sqrt{V^2(t) + g}}{V(t_0) + \sqrt{V^2(t_0) + g}} + a_{22} z_0^2 \frac{V(t_0) + \sqrt{V^2(t_0) + g}}{V(t) + \sqrt{V^2(t) + g}}.
\] (26)
It is clear that \( \forall V(t) \) the magnitude \( V(t) + \sqrt{V^2(t) + g} > 0 \). Therefore, we can define magnitudes

\[
-p = \frac{a_{11} y_0}{V(t_0) + \sqrt{V^2(t_0) + g}} < 0, \quad s = a_{22} z_0^2 \cdot (V(t_0) + \sqrt{V^2(t_0) + g}) > 0.
\]

Then Eq. (26) takes the form \( V = -p \cdot (V + \sqrt{V^2 + g}) + s \cdot (V + \sqrt{V^2 + g})^{-1} \). From here it follows that

\[
V = \pm \frac{s - pg}{(1 + 2p)(1 + 2p + 2s - 2pg)}.
\]

Thus, the solvability condition of Eq. (26) is defined by the condition

\[
1 + 2p + 2s - 2pg > 0.
\] (27)

Let condition (27) be fulfilled. Then from (24) it follows that

\[
\exp \left( \int_{t_0}^{t} x(\tau) d\tau \right) = \frac{-V + \sqrt{V^2 + g}}{-2a_{11} y_0} > 0.
\]

As the function \( V(t) \) is bounded, then the function \( \int_{t_0}^{t} x(\tau) d\tau \) as \( t \to \infty \) is also bounded. Hence, it have to be \( \lim_{t \to \infty} x(t) = 0 \) (the convergence condition of an improper integral) and \( \lim_{t \to \infty} x(t) = 0 \).

Introduce into system (23) new variables \( \rho \) and \( \phi \) under the formulas: \( y = \rho \cos \phi, z = \rho \sin \phi \), where \( \rho > 0 \). Then, we get

\[
\begin{align*}
\dot{x}(t) &= a_1 \cdot x(t) + \rho \cdot x(t) \cdot (a_{11} \cos^2 \phi(t) + a_{12} \cos \phi(t) \sin \phi(t) + a_{22} \sin^2 \phi(t)), \\
\dot{\rho}(t) &= (b_1 + (b \cos^2 \phi(t) + c \sin^2 \phi(t)) \cdot x(t)) \cdot \rho(t), \\
\dot{\phi}(t) &= -c_1 - (b - c) \cos \phi(t) \cdot \sin \phi(t) \cdot x(t). 
\end{align*}
\] (28)

From the first equation of system (28) it follows that the equation \( V(t) \equiv \ddot{x}(t) - a_1 x(t) = 0 \) has a countable number of roots \( t_1, \ldots, t_n, \ldots \), where \( \phi(t_i) = k \pi; k_i \) is an integer. It means that the function \( V(t) \) changes a sign with period \( \pi \). Hence, functions \( x(t) \) and \( \dot{x}(t) \) change also signs with that period \( \pi \).

It is clear that there exists \( t^* \in [0, \infty) \) such that at \( t > t^* \) the function \( |x(t)| \) is decreasing. Fig. 6 demonstrates this assertion.

At last if \( x(t) \to 0 \), then at \( b_1 = c_1 = 0 \) and by virtue of convergence of integral \( \int_{t_0}^{t} x(\tau) d\tau \) from the second and third equations of system (23) it follows that solutions \( y(t) \) and \( z(t) \) are bounded.

(c) We rewrite system (23) in the form:

\[
\dot{x} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & b_1 & c_1 \\ 0 & -c_1 & b_1 \end{pmatrix} x + G(x) \equiv A x + G(x) \equiv F(x), \quad x = (x, y, z)^T, \quad G, F : \mathbb{R}^3 \to \mathbb{R}^3.
\]

![Fig. 6. The evolution of the function \( x(t) \).](image-url)
Assume that there exists a moment $t$.

**Theorem 5.** From system (28) we have:

\[
x(t) = \exp(At)x_0 + \int_0^t \exp(A(t - \tau))G(x(\tau))d\tau,
\]

where $x_0 \in W$.

From (29) it follows that

\[
\|x(t)\| \leq k\|x_0\| + \int_0^t k\|G(x(\tau))\|d\tau,
\]

where $k = \|\exp(At)\| < \infty$.

At $x_0 \in W$ the solution $x(t)$ of the equation $x = G(x)$ is bounded. Hence, we have to have $\int_0^\infty \|G(x(\tau))\|d\tau < \infty$. Therefore, on the basis of Gronwall–Bellman inequality from (30), we derive

\[
\|x(t)\| \leq k\|x_0\| \exp \left[ \int_0^\infty k\|G(x(\tau))\|d\tau \right] < \infty.
\]

Theorem is proved. □

Let $x(t_i) = x_i$, $\rho(t_i) = \rho_i$, $\phi(t_i) = \phi_i$, where $t_i$ are roots of the first equation $\dot{x}(t_i) = 0$ of system (28), $i = 1, 2, \ldots$. We can consider that $\cdots < \phi_{i-1} < \phi_i < \phi_{i+1} < \cdots$.

We will assume that one of the following two variants takes place:

(a) either $t_i$, $t_{i+1}$, and $t_{i+2}$ are sequential a maximum, minimum, and maximum of function $\rho(t)$; $i = 1, 2, \ldots$;

(b) or $t_i$, $t_{i+1}$, and $t_{i+2}$ are sequential a maximum, minimum, and maximum of function $x(t)$; $i = 1, 2, \ldots$.

Assume that the case (b) takes place.

**Theorem 5.** Assume that there exists a moment $t'$ such that from the condition $t > t'$ it follows that $x(t) \geq 0$. Let $\forall i \in \{1, 3, 5, \ldots \}$ the condition

\[
\int_{t_i}^{t_{i+2}} \left[ b + c + (b - c) \cos 2\phi(\tau) + \frac{b_1(b - c)}{c_1} \sin 2\phi(\tau) \right] \cdot x(\tau)d\tau < 0.
\]

be fulfilled. Then under the conditions of Theorem 4 in system (23) there are either limit cycles or chaotic attractors.

**Proof.** From system (28) we have:

\[
x(t_i) = -\rho^2(t_i) \cdot (a_{11} \cos^2 \phi(t_i) + a_{12} \cos \phi(t_i) \sin \phi(t_i) + a_{22} \sin^2 \phi(t_i)),
\]

\[
i = 1, 2, \ldots
\]

Consider the fraction

\[
x_i = \frac{\rho_i^2 \cdot (a_{11} \cos^2 \phi_{i+1} + a_{12} \cos \phi_{i+1} \sin \phi_{i+1} + a_{22} \sin^2 \phi_{i+1})}{\rho_i^2 \cdot (a_{11} \cos^2 \phi_i + a_{12} \cos \phi_i \sin \phi_i + a_{22} \sin^2 \phi_i)}.
\]

From (32) it follows that $\forall i$ the magnitude $\phi(t_{i+2}) - \phi(t_i) = \phi(T)$, where $\phi(T)$ is a period of the function $a_{11} \cos^2 \phi(t) + a_{12} \cos \phi(t) \sin \phi(t) + a_{22} \sin^2 \phi(t)$.

Then

\[
\frac{a_{11} \cos^2 \phi_{i+2} + a_{12} \cos \phi_{i+2} \sin \phi_{i+2} + a_{22} \sin^2 \phi_{i+2}}{a_{11} \cos^2 \phi_i + a_{12} \cos \phi_i \sin \phi_i + a_{22} \sin^2 \phi_i} = 1.
\]

Any function of the kind $d_{11} \cos^2 \phi + d_{12} \cos \phi \sin \phi + d_{22} \sin^2 \phi$ always can be transformed to the form $g_1 + g_2 \cos 2\phi + g_{12} \sqrt{1 - \cos^2 2\phi}$. Hence, the period of these functions is equal $\pi$. Then from the third equation of system (28), we have

\[
\phi(t_{i+2}) - \phi(t_i) = \pi = -c_1(t_{i+2} - t_i) - \frac{b - c}{2} \int_{t_i}^{t_{i+2}} \sin 2\phi(t) x(t)dt.
\]

Therefore, the fraction $x_{i+2}/x_i$ may be rewritten as

\[
x_{i+2}/x_i = \exp \left[ \frac{-2b_1 \pi}{c_1} + \int_{t_i}^{t_{i+2}} \left[ b + c + (b - c) \cos 2\phi(\tau) - \frac{b_1(b - c)}{c_1} \sin 2\phi(\tau) \right] \cdot x(\tau)d\tau \right]
\]

\[= \lambda \exp \left[ \int_{t_i}^{t_{i+2}} \left[ b + c + (b - c) \cos 2\phi(\tau) - \frac{b_1(b - c)}{c_1} \sin 2\phi(\tau) \right] \cdot x(\tau)d\tau \right],
\]
where $\lambda = \exp (-2b_1\pi/c_1)$.

Let's introduce the function

$$h(\phi) = b + c + (b - c) \cos 2\phi - \frac{b_1(b - c)}{c_1} \sin 2\phi.$$  

The bounded function $x(t)$ is a monotone decreasing on interval $[t_i, t_{i+1}]$, and it is a monotone increasing on interval $[t_{i+1}, t_{i+2}]$. Then we have (a theorem about average value):

$$\int_{t_i}^{t_{i+2}} h(\phi(t)) \cdot x(t) \, dt = \int_{t_i}^{t_{i+1}} h(\phi(t)) \cdot x(t) \, dt + \int_{t_{i+1}}^{t_{i+2}} h(\phi(t)) \cdot x(t) \, dt$$

$$= x(t_i + 0) \int_{t_i}^{t_i} h(\phi(t)) \, dt + x(t_{i+1} - 0) \int_{t_{i+1}}^{t_{i+1} + 0} h(\phi(t)) \, dt + x(t_{i+1} + 0) \int_{t_{i+1} + 0}^{t_{i+2}} h(\phi(t)) \, dt$$

$$+ \int_{t_{i+1}}^{t_{i+2}} h(\phi(t)) \, dt + x(t_{i+2} - 0) \int_{t_{i+2} - 0}^{t_{i+2}} h(\phi(t)) \, dt,$$

where $t_i \leq \xi \leq t_{i+1}, t_{i+1} \leq \zeta \leq t_{i+2}$. We can consider that $t^* < t_0$. Then function $x(t)$ is positive on interval $[t_r, t_{i+2}]$. Therefore, there exist the magnitudes $\xi$ and $\zeta$ such that

$$x(t_{i+1} - 0) \int_{\xi}^{t_{i+1}} h(\phi(t)) \, dt = x(t_{i+1} + 0) \int_{t_{i+1}}^{\zeta} h(\phi(t)) \, dt = 0.$$

Hence, from (33) it follows that

$$\int_{t_i}^{t_{i+2}} h(\phi(t)) \cdot x(t) \, dt = p_i x_i + p_{i+2} x_{i+2},$$

where magnitudes $p_i = \int_{t_i}^{t_i} h(\phi(t)) \, dt, p_{i+2} = \int_{t_{i+2}}^{t_{i+2}} h(\phi(t)) \, dt$ can have any signs.

Finally, we get

$$x_{i+2} = \lambda x_i \exp(p x_i + p_{i+2} x_{i+2}): \quad i = 1, 3, \ldots, 2n - 1, \ldots \quad (34)$$

It is easy to check that if $\forall p_i = -p, p_{i+2} = p$, then system (23) has either periodic or almost-periodic solutions. Really, in this case from (34) it follows that $x_{i+2} = \lambda^2 x_i \exp(-p x_i + p x_{i+2})$. If $\lambda = 1$, then a solution of system (23) is periodic; if $\lambda \neq 1$, then a solution will be almost-periodic.

It is obvious if $p x_i + p_{i+2} x_{i+2} < 0$, then condition (31) is hold. Suppose that $p_i = -p < 0, p_{i+2} = -m < 0$, and $x_{i+2} = u(x_i, x_{i+2}) = \lambda x_i \exp(-p x_i - m x_{i+2})$; $i = 1, 3, \ldots, 2n - 1, \ldots$. Let's compute sequential iterations in process (34):

$$x_1 = u(x_1, x_3) = \lambda x_1 \exp(-p x_1 - m x_3);$$

$$x_3 = u(x_3, x_5) = \lambda^2 x_3 \exp(-p x_3 - m x_5) \cdot \exp(-p \lambda x_3 \exp(-p x_3 - m x_5) - m x_5);$$

$$\cdots \cdots \cdots$$

$$x_{2n-1} = u(\ldots u(x_{2n-3}, x_{2n-1})).$$

In order to compute fixed points of 1-dimension mappings $u(x_1, x_3), \ldots, u(\ldots u(x_1, \ldots, x_{2n-3}, x_{2n-1})$ on every iteration of process (34), in Eqs. (35) we assume $x = x_1 = \cdots = x_{2n-1}$. Then the sequence of equations with respect to one variable $x$ will have the form:

$$x = \lambda x \exp(-p x + m x).$$

It is easy to check that in the process

$$x_{i+2} = \lambda x_i \exp(-p x_i): \quad i = 1, 3, \ldots, 2n - 1, \ldots \quad (37)$$

equations for finding of fixed points of every iteration have a structure the same as Eqs. (36). Thus, if $p_i < 0, p_{i+2} < 0, x_i > 0$, and $x_{i+2} > 0$, then iterated process (34) is similar to process (37). (In this case the conditions of Theorem 5 are fulfilled.)

Consider the function $w = f(\mu) = \lambda x \exp(-\mu z)$, where $\lambda > 0, \mu > 0, x \in [0, \infty)$. We have to show that the mapping $f(z) : [0, \infty) \rightarrow (0, \infty]$ beginning with some $\lambda$ and $\mu$ is chaotic. (For example, let $\lambda = 20, \mu = 2, x \in [0, \infty).$ On Figs. 7, 8 two sequential iterations of the function $w = f(z) = f(\mu)$ for the given $\lambda$ and $\mu$ are represented.)

Let's change the variable $z$ in the function $w = f(z) : v = \mu z$. Then we will have $w = (\lambda z) v \exp(-v)$. Notice that the function $(\lambda z) v \exp(-v)$ has a unique maximum $\lambda z |e(\mu z)$ in the point $v = 1$ (here $e = 2.71828 \ldots$). From here it follows that this function is unimodal.

Consider the iteration process $x_{i+1} = r x \exp(-x_i)$, where $r > 0$. Define the function $h(v) = r \exp(-v)$. It is clear that the inverse mapping $h^{-1}(v)$ has two branches: $h_1^{-1}(v)$ and $h_2^{-1}(v)$, where each of the mappings $h_1^{-1}(v)$ and $h_2^{-1}(v)$ is invertible.

Assume

$$H(v) = h_1^{-1} \left( h_2^{-1} \left( \ldots \left( h_{k}^{-1}(v) \right) \right) \right), \quad k = 2, 3, \ldots$$

where either $h_k = 1$ or $h_k = 2$. It is clear that the mappings $H(v)$ is monotonically.
The mapping $H(v)$ has a fixed point $v^\star$. It is known that the fixed point $v^\star$ of the mapping $H(v)$ corresponds to a fixed point $v^\star$ of the mapping $h^{(k)}(v) = h(h(\ldots h(v)))$. The point $v^\star$ of the mapping $h^{(k)}(v)$ corresponds to either a fixed point or a $p$-limit cycle of the mapping $h(v)$. Since a positive integer $k$ may be an arbitrary and $h_1^{-1}(h_2^{-1}(v)) \neq h_2^{-1}(h_1^{-1}(v))$, then the nonmonotone function $h(v)$ can have any number of cycles of different multiple $p$ and any number of nonperiodic trajectories.

It is known that any nonmonotone 1-dimension mapping of the type of represented on Fig. 7 (for example $z_{i+1} = rz_i(1 - z_i)$, $r > 0$; $i = 1, 2, \ldots$) generates a chaotic attractor. Therefore, the iterated process (34) $\forall x > 0$, $\forall p, q, p_i > 0$, and $\forall x_i > 0$ generates in system (28) (or (23)) a chaotic attractor. 

To build an explicit procedure for process (34) is impossible. However on every step of iterated process (34) finding all fixed points is possible. For example, assume $\lambda = 20, p_1 = -4, p_{i+2} = -1$. Then after the first iteration, we have the equation $x = \lambda \exp ((p_1 + p_{i+2})x)$; this equation has two roots: 0, 0.6. After the second step, we get the equation $x = \lambda^2 \exp((p_1 + 2p_{i+2})x + p_2 \exp ((p_1 + p_{i+2})x))$; this equation has four roots: 0, 0.11, 0.6, 0.82 and so on. Finally, we derive the Feigenbaum scenario of doubling of the period.
Let \( x_i = z_i > 0, p_i = -1, p_{i+2} = 0, x_{i+2} = z_{i+1} \). Then process (34) may be rewritten as:

\[
    z_{i+1} = z_i \exp(r - z_i), \quad r = -\frac{2\pi b_1}{c_1} > 0, \quad i = 0, 1, 2 \ldots
\]  

(38)

Process (38) is called the Ricker discrete population model [27,28]. It is known [26] that the Ricker model has 2-cycle, 4-cycle and chaotic attractors when \( r = 2.1, r = 2.6 \) and \( r = 3 \), respectively.

5. Chaotic attractors of nonhomogeneous quadratic systems: case \( b + c \neq 0 \)

**Theorem 6.** Assume \( b + c \neq 0 \) and there exists a moment \( t^* \) such that from the condition \( t > t^* \) it follows that the solution \( x(t) \) of system (23) is bounded and positive. Let conditions \( a_1 > 0, b_1 < 0, \) and (31) be also valid. Then under the conditions of Theorem 1 in system (23) there are either limit cycles or chaotic attractors.

**Proof.** It is always possible to get \( c_1 > 0 \) by replacements of variables \( y \to -y, z \to -z \). Therefore, we can consider that in (28) \( c_1 > 0 \). Then from (28), we have

\[
    \rho(t) = \rho_0 + \int_0^t \left( |b_1 + (b \cos^2 \phi(t) + c \sin^2 \phi(t)) \cdot x(t)| \cdot \rho(t) \right) dt < \rho_0 + \int_0^t \left( |b \cos^2 \phi(t) + c \sin^2 \phi(t)| \cdot x(t) \cdot \rho(t) \right) dt.
\]  

(39)

Fig. 9. The graph of dependence \( x(t) \) on \( \phi(t) \) for system (42); here \( t = 4.5 - 7.2 \).

Fig. 10. The projection of the chaotic attractor of system (42) onto the plane \( XOY \) for \( t = 15 \).
The solution $x(t)$ is bounded. From (39) it follows that under the conditions of Theorem 1 the solution $p(t)$ is bounded (we remind that $b_1 < 0$). It means that solutions $y(t)$ and $z(t)$ of system (23) is also bounded.

If we postulate an boundedness of the solution $x(t)$, then a proof of Theorem 6 fully repeats the proof of Theorem 5 at the condition $b + c \neq 0$. □

6. Examples

Let’s compute equilibriums of system (23):

\[
\begin{align*}
    a_1x + a_{11}y^2 + a_{12}yz + a_{22}z^2 &= 0, \\
    (b_1 + bx)y + c_1z &= 0, \\
    -c_1y + (b_1 + cx)z &= 0.
\end{align*}
\]  

It is clear that for existence of nontrivial equilibriums the condition $bcx^2 + (b + c)b_1x + b_1^2 + c_1^2 = 0$ must be valid. From here we get

\[
x_{1,2} = \frac{-(b + c)b_1 \pm \sqrt{(b - c)^2b_1^2 - 4bcc_1^2}}{2bc}.
\]

![Fig. 11. The projection of the chaotic attractor of system (42) onto the plane YOZ for $t = 15$.](image1)

![Fig. 12. The projection of two coexisting chaotic attractors of system (42) onto the plane YOZ for $t = 15$.](image2)
Let $bc \neq 0$, $a_1 > 0$, $b_1 < 0$, $a_{12} = 0$, $a_{11} < 0$, $a_{22} > 0$. Then eigenvalues of the Jacobi matrix of system (23) are $(x_1, y_1, z_1) = (x_1, -c_1 x_1 (b + b_1 x_1) x_1)$ and $(x_2, y_2, z_2) = (x_2, -c_1 x_2 (b + b_2 x_2) x_2)$, where $x$ is a root of the equation:

$$a_1 x_1^2 + a_2 x_1^2 (b + b_1 x_1) = 0. \quad (41)$$

If $(b - c)^2 b_1 - 4bcc_1^2 < 0$, then system (40) has one root $(0, 0, 0)$.

If $(b - c)^2 b_1 - 4bcc_1^2 = 0$ and Eq. (41) has two solutions (for $x_1 = -(b + c)/(2bc)$) with respect to $x$, then system (40) has three roots (including $(0, 0, 0)$).

If $(b - c)^2 b_1 - 4bcc_1^2 > 0$ and Eq. (41) has two solutions (only either for $x_1$ or only for $x_2$) with respect to $x$, then system (40) has three roots (including $(0, 0, 0)$).

If $(b - c)^2 b_1 - 4bcc_1^2 > 0$ and Eq. (41) has four solutions (for $x_1$ and $x_2$) with respect to $x$, then system (40) has five roots (including $(0, 0, 0)$).

Fig. 13. The projection of the couple limit cycles of system (43) onto the plane XOY for $t = 15$.

Fig. 14. The graph of dependence $x(t)$ on $f(t)$ for system (43); here $t = 4.5 - 4.9$. 
1. Consider the system

\[
\begin{align*}
\dot{x}(t) &= 4x(t) - y^2(t) + z^2(t), \\
\dot{y}(t) &= -15y(t) - 70z(t) + 15x(t)y(t), \\
\dot{z}(t) &= 70y(t) - 15z(t) - 15x(t)z(t).
\end{align*}
\]  

(42)

(Here \( b + c = 0 \)).

The equilibriums in this case are:

\( O_1(0,0,0) \); \( O_2(-4.7726, 6.0008, -7.4230) \); \( O_3(-4.7726, -6.0008, 7.4230) \); \( O_4(4.7726, -7.423, -6.0008) \); \( O_5(4.7726, 7.423, 6.0008) \).

The eigenvalues of the Jacobi matrix in points \( O_1, O_2, O_3, O_4, \) and \( O_5 \) are:

\( (4, -15 + 70i, -15 - 70i) \); \( (2.0011 + 52.2420i, 2.0011 - 52.2420i, 2.0011 + 52.0954i, -30.0022) \); \( (2.0011 - 52.2420i, 2.0011 + 52.2420i, -2.0011 - 52.0954i, -30.0022) \); and \( (2.0011 + 52.2420i, 2.0011 - 52.2420i, 2.0011 + 52.0954i, -30.0022) \). All equilibriums are saddle-foci.

From Fig. 9 it follows that \( \phi_1 = 88.80, \phi_{i+2} = 91.94. \) Therefore, \( \phi_{i+2} - \phi_1 = \pi = 3.14 \), and for (34) we have \( p_i < 0, p_{i+2} < 0, x_i > 0, \) and \( x_{i+2} > 0. \) Thus, all conditions of Theorem 5 are fulfilled.
Under the conditions in system (42) there are two Lorenz-type chaotic attractors located in half-spaces  \( x > 0 \) and  \( x < 0 \), and turned with respect to each other on  \( \pi/2 \). On Figs. 10–12 different projections of these attractors are represented.

2. Consider the system

\[
\begin{align*}
\dot{x}(t) &= 4x(t) - y(t)^2 + z(t), \\
\dot{y}(t) &= -5y(t) - 70z(t) + 15x(t)y(t), \\
\dot{z}(t) &= 70y(t) - 5z(t) - 15x(t)z(t).
\end{align*}
\]  \quad (43)

(Here  \( b + c = 0 \).)

System (43) has five equilibriums of the same type that and system (42). These equilibriums are located symmetric with respect to origin of coordinates and the coordinate plane \( XOY \). On Fig. 13 are shown that system (43) have two limit cycles. (Fig. 14 confirms the existence of a periodic solution.)

Let's consider that for system (23) \( a_1 = 4, c_1 = -70, a_{11} = -1, a_{12} = 0, a_{22} = 1, b = 15, c = -15 \). If  \(-14.1 < b_1 < -2.1\), then any trajectory of this system is periodic. If  \(-32.2 < b_1 < -14.1\), then any trajectory of this system is chaotic.

3. Now consider the system

![Figure 17](image17.png)

**Fig. 17.** The projection of the chaotic attractor of system (44) onto the plane \( YOZ \) for  \( t = 15 \).

![Figure 18](image18.png)

**Fig. 18.** The projection of two chaotic attractors of system (44) onto the plane \( YOZ \) for  \( t = 15 \).
\[
\begin{aligned}
\dot{x}(t) &= 4x(t) - y^2(t) + z^2(t), \\
\dot{y}(t) &= -15y(t) - 70z(t) + 10x(t)y(t), \\
\dot{z}(t) &= 70y(t) - 15z(t) - 15x(t)z(t).
\end{aligned}
\] (44)

(Here \(b + c \neq 0\).

The equilibriums in this case are \(O_1(0,0,0)^T; O_2(8.1821, 9.7145, -7.8513)^T; O_3(8.1821, -9.7145, 7.8513)^T\). The eigenvalues of the Jacobi matrix in points \(O_1, O_2,\) and \(O_3\) are \((4, -15 + 70i, -15 - 70i), (2 + 61.0954i, 2 - 61.0954i, -42.9114),\) and \((2 - 61.0954i, 2 + 61.0954i, -42.9114)\).

We will not demonstrate projections of the attractor on coordinate planes \(XOY, YOZ\) because they look like at the same projections on Figs. 10, 11. On Figs. 16, 17 are shown different characteristics of the second attractor of system (44). On Fig. 18 the projection on coordinate plane \(YOZ\) of both attractors of system (44) is represented.

Notice that the points \(\phi_i, \phi_{i+2}, i = 1,3,5,\ldots\) are exactly the roots of \(\pi\)-periodic function \(h(\phi)\). It is possible to check that all conditions of Theorem 6 are fulfilled. (It is visible on Fig. 15; here \(p_i < 0, p_{i+2} < 0, \phi_i > 0,\) and \(\phi_{i+2} > 0\.)

4. Consider the system

\[
\begin{aligned}
\dot{x}(t) &= 4x(t) - y^2(t) + z^2(t), \\
\dot{y}(t) &= -18y(t) - 70z(t) + 10x(t)y(t), \\
\dot{z}(t) &= 70y(t) - 18z(t) - 15x(t)z(t).
\end{aligned}
\] (45)

Fig. 19. The projection of the coexisting limit cycle and chaotic attractor of system (45) onto the plane \(XOY\) for \(t = 15\).

Fig. 20. The projection of attractor evolutions of system (46) onto the plane \(XOZ\) for \(t = 3\).
System (45) has five equilibriums of the same type that and system (42) or (43) (all equilibriums are saddle-foci). In system (45) there is one limit cycle and one Lorenz-type attractor. These attractors are shown on Fig. 19.

At last we consider the system

\[
\begin{align*}
\dot{x}(t) &= -x(t) - y^2(t) - 15z^2(t), \\
\dot{y}(t) &= 45y(t) + 160z(t) + x(t)y(t), \\
\dot{z}(t) &= -160y(t) + 45z(t) + 15x(t)z(t).
\end{align*}
\]

System (46) has a single equilibrium (0,0,0) which is a saddle-focus. In this system there is either a limit cycle or a sphere-shaped chaotic attractor. Evolutions of the chaotic attractor at initial values \(x_0 = 0, y_0 = 0, z_0 = 0.1\) are shown on Figs. 20, 21.

From Figs. 20, 21 it is visible that the trajectory evolves between planes \(x = -12\) and \(x = 0\). The sphere-shaped attractor is disposed in a neighborhood of point \((-6,0,0)\).

As the final result it should be note that in system (23) can be realized one of the following geometric structures: two Lorenz-type attractors; two limit cycles; one Lorenz-type attractor and second chaotic attractor of a new type; one Lorenz-type attractor and one limit cycle; either one sphere-shaped chaotic attractor or one limit cycle.

References