Temperature Models for Ware Hall

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Introduction

In August, our mathematics department moved into the former music building, Ware Hall. The university renovated the old choir rehearsal hall to form four faculty offices and one computer lab, as in Figure 1. The temperature of all of these rooms is controlled by a single thermostat, which the renovation enclosed in a faculty office.

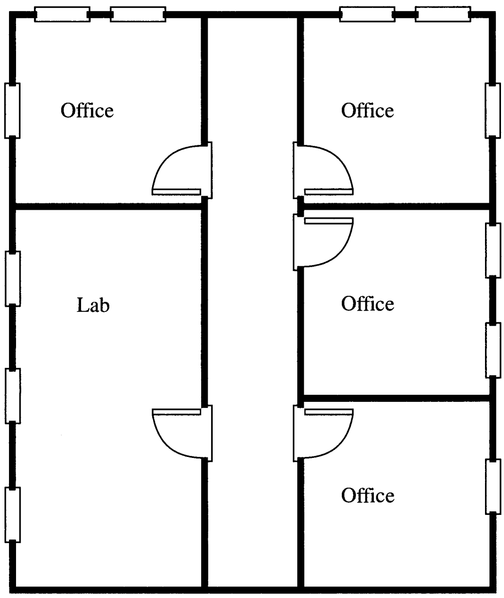


Figure 1. The renovated choir rehearsal hall now holds four faculty offices and one computer lab.

With this system, our students literally sweat out their calculus labs as we sit in our arctic offices, teeth chattering. In this paper we use elementary differential equations to investigate the temperature issues associated with our heating, ventilation, and air conditioning (HVAC) unit, beginning with a straightforward and idealized model and moving to more complicated, realistic models.

Model 1: Newton's law of heating and cooling

A number of factors affect the temperature in our computer lab: the outdoor temperature, the volume of output of the HVAC unit, the temperature of the air circulated by the HVAC unit, the size of the lab room, the number of people in the room, and the number of computers running in the room. We begin by focusing only on the first factor—the outdoor temperature. Ignoring all other sources of hot or cold air, we may treat the change in lab temperature due to outdoor temperature as a Newton's Law of Heating and Cooling problem. That law states that the rate of change of the temperature of an object is directly proportional to the difference between the object's temperature and the temperature of the surrounding medium. That is,

$$\frac{dT}{dt} = k\,(T_a - T), \tag{1}$$

where T_a is the outdoor, ambient temperature, and T is the temperature of the object, which is the air in the lab room. This model assumes that the constant k will account for the insulation surrounding the room and the insulation of the windows and that all doors and windows in the room are closed. We solve this differential equation using separation of variables to obtain

$$T(t) = T_a + (T_0 - T_a)\,e^{-kt}, \tag{2}$$

where T_0 is the initial temperature of the room. As can be seen in both Equations (1) and (2), equilibrium occurs when the initial temperature and the ambient temperature are the same. The graph in Figure 2 shows that, when $T_0 > T_a$ or $T_0 < T_a$, the solution approaches T_a exponentially as time increases. Note that this model treats T_a as a constant, even though outdoor temperature typically varies over the course of the day. This assumption can be eliminated by replacing T_a in (1) with a function of t that fits data for outdoor temperature changes, as done in [**3**].

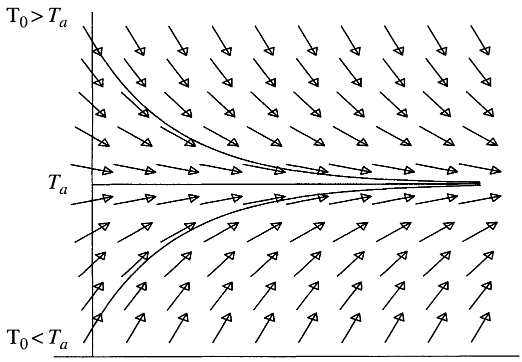


Figure 2. When $T_0 > T_a$ or $T_0 < T_a$, the solution approaches T_a exponentially as time increases.

Model 2: A mixing problem

To have the air temperature of an office or computer lab approach the outdoor temperature is quite undesirable during the blistering heat of a Georgia summer. Of course, in the summer, we turn on the HVAC system, which has a thermostat inside the lab room and keeps the temperature within $\pm w$ degrees Fahrenheit of a set temperature T_s. The HVAC system runs when the temperature is at or above $T_s + w$ until the air temperature in the room reaches $T_s - w$. Likewise, the system is off when the temperature is at or below $T_s - w$ until the air temperature reaches $T_s + w$. Thus, the room temperature stays between $T_s - w$ and $T_s + w$, and the HVAC system does not continually switch on and off in an attempt to maintain a single, constant room temperature.

We now extend our first model to describe the interplay of the warming from outdoors with the cooling produced by the air-conditioned air when the HVAC system is on. The new differential equation is

$$\frac{dT}{dt} = k\,(T_a - T) + \frac{r}{V}\,(T_{\text{hvac}} - T), \tag{3}$$

where T_{hvac} is the temperature of cooled air pumped out by the HVAC system, r is the rate (in liters per minute) at which the air is pumped out, and V is the volume of the lab. The first summand comes from Newton's Law of Heating and Cooling and describes the effect of the warm outdoors. The second summand comes from considering room temperature as concentration (for example, as proportional to calories per liter) and writing the differential equation for a mixing problem. We have assumed that the rate at which cool air enters the room is the same as the rate at which air leaves the room and that the air in the room remains completely mixed. This model comes from the standard mixing problems taught in elementary differential equations courses [**1, 5**].

By rewriting equation (3) as

$$\frac{dT}{dt} = \left(k + \frac{r}{V}\right)\left(\frac{VkT_a + rT_{\text{hvac}}}{Vk + r} - T\right), \tag{4}$$

we see that the equilibrium temperature that produces $\frac{dT}{dt} = 0$ is

$$T = \frac{VkT_a + rT_{\text{hvac}}}{Vk + r}. \tag{5}$$

We call this value m_{krV}. Given any starting temperature, the air condition system will cause the room temperature to approach m_{krV} degrees.

Example 1. *Measurements show that the lab room has volume* 204,000 L, *and the physical plant reports that the HVAC unit is designed to pump air into the lab at a rate of* 26,000 L/min. *To simulate summer conditions in Georgia, let* $k = .0248/\text{minute}$, *outdoor temperature* $T_a = 90°\text{F}$, *initial temperature* $T_0 = 78°\text{F}$, *HVAC air temperature* $T_{\text{hvac}} = 60°\text{F}$, *and set temperature* $T_s = 70°\text{F}$ *with* $w = 3°\text{F}$. *The differential equation is*

$$\frac{dT}{dt} = 0.1523\,(64.89 - T).$$

The graph in Figure 3 shows an asymptote at the equilibrium temperature of $T = m_{krV} = 64.89°\text{F}$.

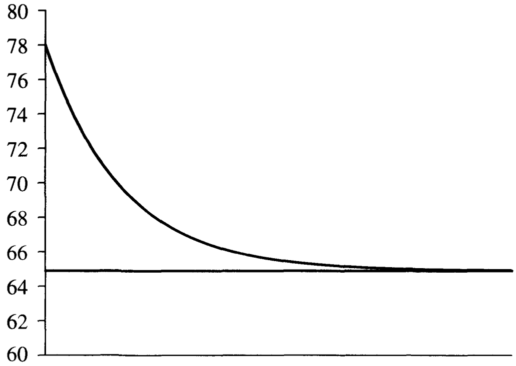


Figure 3. Using the parameters for Example 1, the solution approaches $m_{krV} = 64.89°\text{F}$ exponentially as time increases.

Separation of variables yields a solution to Equation (4). When the HVAC system is on, the room temperature is decreasing ($\frac{dT}{dt} < 0$), and the model equation is

$$T(t) = m_{krV} + (T_0 - m_{krV})\, e^{-(k+\frac{r}{V})t}. \tag{6}$$

When the HVAC is on ($\frac{dT}{dt} < 0$ and $T > T_s - w$), one important consideration is whether the temperature in the room will ever reach $T_s - w$ so that the HVAC will turn off. In order to reach this switch-off temperature, T_{hvac} must be small enough to ensure that the equilibrium point for equation (4) is below $T_s - w$; otherwise, the temperature will approach the switch-off temperature but never reach it, as in Figure 3. This requires that

$$m_{krV} = \frac{VkT_a + rT_{\text{hvac}}}{Vk + r} < T_s - w,$$

and so the HVAC must output air whose temperature is

$$T_{\text{hvac}} < (T_s - w) - \frac{Vk}{r}\big(T_a - (T_s - w)\big).$$

The expression shows T_{hvac} must be less than the lower set temperature, $T_s - w$, minus a value proportional to the difference between the outdoor temperature and $T_s - w$. For Example 1, T_{hvac} must be less than 62.52°F, which is significantly lower than the desired temperature range of $70° \pm 3°\text{F}$. This explains why the air blowing out of air conditioning systems needs to be significantly cooler than the desired room temperature and why you often need a sweater when sitting in front of an air conditioning duct in the summer!

Another consideration is how long the air conditioning system must run to cool the room to $T_s - w$ and how long the room will take to warm up to $T_s + w$ when the air conditioning is off. When the air conditioning is on, the room temperature is described by Equation (6). Finding the time necessary to cool the room from $T_s + w$ to $T_s - w$ can be accomplished by first setting $T_0 = T_s + w$ so that $t = 0$ corresponds to the air conditioner kicking on. Then set the equation equal to $T_s - w$, and solve for t to find the cooling time as

$$t_{\text{cool}} = -\frac{V}{Vk + r} \ln\left(\frac{T_s - w - m_{krV}}{T_s + w - m_{krV}}\right). \tag{7}$$

Similarly, from equation (2), the time needed for the room to warm up to $t_s + w$ is

$$t_{\text{warm}} = -\frac{1}{k}\ln\left(\frac{T_s + w - T_a}{T_s - w - T_a}\right). \tag{8}$$

For Example 1, $t_{\text{cool}} = 8.84$ minutes and $t_{\text{warm}} = 12.19$ minutes. These calculations emphasize that, after the temperature reaches the range $T_s - w$ to $T_s + w$, the solution is periodic. The increasing intervals correspond to a piece of the graph from Figure 2, because the only temperature change during these times is due to Newton's Law of Heating and Cooling. The decreasing portions are described by Equation (3) and correspond to a piece of Figure 3.

The models in Equations (1) and (4) can now be combined to form a single differential equation that describes the temperature of the lab room with an HVAC system. By using a step function that only takes on the values 0 and 1 to turn the mixing term in the differential equation on and off, we can simulate the feedback thermostat that turns the HVAC system on and off. This behavior of the thermostat is slightly tricky to model with an equation, since the term describing the effect of the air conditioning must be off if the room temperature is increasing and below $T_s + w$ but on if the room temperature is decreasing and above $T_s - w$. The sign of $\frac{dT}{dt}$, denoted $\text{sgn}(\frac{dT}{dt})$, plays an important role. Since step functions switch on or off at only one temperature, we use the following modification to switch at both $T_s - w$ and $T_s + w$:

$$\text{step}(T) = \begin{cases} 1, & \text{if } T > T_s + \text{sgn}(\frac{dT}{dt}) \cdot w \\ 0, & \text{otherwise} \end{cases}$$

If the temperature of the room is rising ($\frac{dT}{dt} > 0$), the step function will switch on ($\text{step}(T) = 1$) when the room temperature reaches $T_s + w$. However, if the room temperature is falling ($\frac{dT}{dt} < 0$), the step function will switch off ($\text{step}(T) = 0$) at the temperature $T_s - w$.

Using the step function above, the model differential equation for our lab room with an HVAC system is

$$\frac{dT}{dt} = k\,(T_a - T) + \text{step}(T)\left(\frac{r}{V}\right)(T_{\text{hvac}} - T). \tag{9}$$

Writing an analytic solution for this equation is ungainly due to the switching of the thermostat. First, let t_1 be the time necessary to reach the end of the first cooling cycle. Then, combine the results from our two previous models to define a function, τ, for times between 0 and $t_{\text{warm}} + t_{\text{cool}}$ minutes so that heating occurs for the first t_{warm} minutes and cooling occurs for the next t_{cool} minutes.

$$\tau(t) = \begin{cases} m_{krV} + (T_s + w - m_{krV})\,e^{-(k+\frac{r}{V})(t - t_{\text{warm}})}, & \text{if } t > t_{\text{warm}} \\ T_a + (T_s - w - T_a)\,e^{-kt}, & \text{if } t \le t_{\text{warm}} \end{cases}$$

An analytic solution consists of first using Equations (2) and (6) as necessary until time t_1 is reached. Then, to produce the periodicity seen in air conditioning performance, use

$$T(t) = \tau\big((t - t_1)\,\text{mod}(t_{\text{warm}} + t_{\text{cool}})\big)$$

for $t > t_1$, where $(t - t_1)\,\text{mod}(t_{\text{warm}} + t_{\text{cool}})$ is the remainder of $(t - t_1)$ divided by $(t_{\text{warm}} + t_{\text{cool}})$.

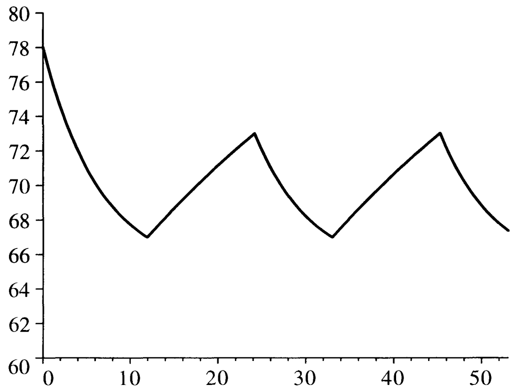


Figure 4. The temperature of an ideal room with an HVAC unit installed, as in Example 1. The time required to cool the room from 73°F to 67°F is $t_{\text{cool}} = 8.84$ minutes, and the time necessary for the room to warm up from 67°F to 73°F is $t_{\text{warm}} = 12.19$ minutes.

Model 3: Two rooms

The university's updates to Ware Hall resulted in the single HVAC thermostat being located in a small faculty office. This office cooled relatively quickly, since there was usually only one faculty member in the office, only one computer in the office, and only one window. The computer lab across the hall, however, regularly held twenty students, ten computers, and one professor. Needless to say, when the HVAC unit had cooled the faculty office adequately, the temperature in the lab was still stifling.

While teaching in the hot lab, our interests quickly turned to modelling the temperatures in the lab and the faculty office. This model is a system of two differential equations in which the HVAC control term in each is dependent on the temperature in the faculty office. For now, we disregard the people and computers in each room. Let F be the temperature of the faculty office, L be the temperature of the lab, and k_1, k_2, k_3, k_4 be constants. T_a and T_{hvac} are defined as before. Moreover, k_2 will be the quotient of the rate at which the HVAC unit pumps air into the office and the volume of the office, and k_4 will be similarly defined for the lab. Then, the office and lab temperatures are given by the system

$$\begin{cases} \dfrac{dF}{dt} = k_1\,(T_a - F) + k_2\,(T_{\text{hvac}} - F)\cdot \text{step}(\chi) \\ \dfrac{dL}{dt} = k_3\,(T_a - L) + k_4\,(T_{\text{hvac}} - L)\cdot \text{step}(\chi), \end{cases} \tag{10}$$

where χ equals F if the thermostat is in the faculty office and χ equals L if the thermostat is in the lab room. Notice that the equations are similar to the second model, except that whether or not the air conditioning is on in each room depends only on the temperature in the room containing the thermostat. In each case, the analytic solution for each room of the system is similar to that in the second model.

Example 2. *To model a lab room and faculty office, assume that the faculty office is approximately half the size of the lab and the lab has two air ducts. Based on our measurements, the faculty office is approximately half the size of the lab, implying* *that the ratios k_2 and k_4 of air flow rate to volume for the lab and the office are the same. This gives the parameters $k_1 = 0.0248$, $k_2 = 0.1275$, $k_3 = 0.0389$, $k_4 = 0.1275$ per minute, ambient temperature $T_a = 90°\text{F}$, set temperature $T_s = 70°\text{F}$, HVAC air temperature $T_{\text{hvac}} = 60°\text{F}$, and $w = 3°\text{F}$. The solution curves for each room are shown in Figure 5.*

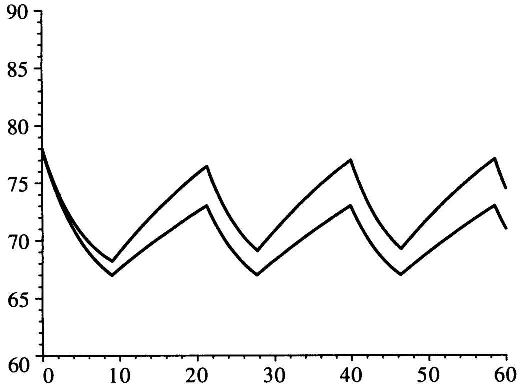


Figure 5. This graph shows the temperatures in the faculty office and the computer lab based on the system (10) using the parameters of Example 2. The system of differential equations was solved using Euler's method with a step size of 0.05 minutes. The upper curve is the lab temperature while the lower curve is the office temperature.

Even in Georgia, we need heat in the winter. The same problems caused by the air conditioning system persist when heating the building. In fact, the same models hold with the modification that the heat is on ($\text{step}(T) = 1$) when $T < T_s + w$ and the temperature is increasing and that the heat is off ($\text{step}(T) = 0$) when $T > T_s - w$ and the temperature is decreasing. This behavior amounts to reflecting our step function about $T = T_s + \text{sgn}(\frac{dT}{dt}) \cdot w$ and can be accomplished with the new step function

$$\text{step}(T) = \begin{cases} 1, & \text{if } T < T_s + \text{sgn}(\frac{dT}{dt}) \cdot w \\ 0, & \text{otherwise.} \end{cases}$$

Model 4: Class is in session

Finally, we create a more realistic model by modifying Equation (10) to include heat from people and computers and to reflect the workings of the HVAC unit in Ware Hall more accurately. The heat gain from one person performing desk work is 475 BTU/hr, and the heat gain from one desktop computer is 1800 BTU/hr [**4**]. Using the lab volume along with the specific heat and density of air, these values give a temperature change of 0.0176°F/min for each person and 0.0668°F/min for each computer in the lab. Since our offices are approximately half of the size of the lab, the temperature change in an office is 0.0352°F/min for each person and 0.1366°F/min for each computer. To account for 21 people in the lab and 10 computers, add $21(0.0176) + 10(0.0668) = 1.0376$°F/min to the differential equation for the lab temperature. Similarly, add 0.1718°F/min to the differential equation for the office temperature.

To further improve our model, we include a modification of the air that is pumped into the rooms by the HVAC unit. The Physical Plant reports that the HVAC unit in

Ware Hall actually recirculates only 80% of the air that it draws out of a room. The other 20% is replaced with air taken from outdoors. There are two intake vents in the lab and one in each of the four faculty offices. Moreover, the cooling unit does not produce air that is at a constant temperature, but pumps out air that has been cooled by 18°F. Given this information, we replace T_{hvac} in the cooling terms with $0.8\,(4F+2L)/6+0.2\,T_a-18$.

The differential equation is now

$$\begin{cases} \dfrac{dF}{dt} = 0.1718 + k_1\,(T_a - F) + k_2\left(\dfrac{4}{15}L + \dfrac{1}{5}T_a - 18 - \dfrac{7}{15}F\right)\cdot \text{step}(F) \\ \dfrac{dL}{dt} = 1.0376 + k_3\,(T_a - L) + k_4\left(\dfrac{8}{15}F + \dfrac{1}{5}T_a - 18 - \dfrac{11}{15}L\right)\cdot \text{step}(F) \end{cases} \tag{11}$$

Its solution is plotted in Figure 6 using the parameters of Example 2.

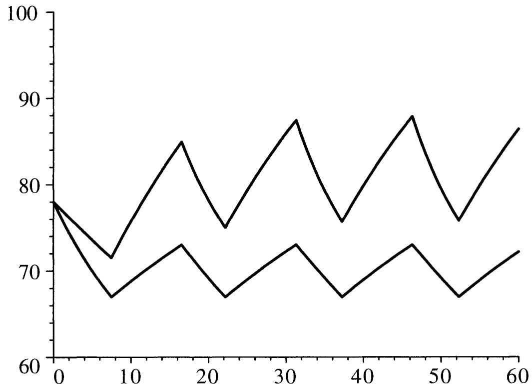


Figure 6. The solution for model 4 is depicted with the upper curve corresponding to the lab temperature and the lower curve corresponding to the office temperature.

Final remarks

The four models presented in this paper give rise to several teaching ideas for use in calculus and differential equations. For example, we have written and successfully used a calculus lab based on this paper. In the lab, students examine slope fields, find the general solution for Newton's Law of Heating and Cooling, calculate cooling and heating times for the lab and office, and explain why the room without the HVAC thermostat has temperature problems. The lab piqued the students' interest and imagination, particularly since they were sitting in the lab room in question. (The lab may be found at http://faculty.mercer.edu/denny_jk/calclab.pdf.)

For differential equations, these models provide opportunities for student projects. Such assignments might include programming Euler's method, investigating slope fields for the models, using a function to model outdoor temperature changes during a typical day (see **[4]** for a set of temperature data), finding the optimum room size for a given number of computers and people, or collecting temperature data and determining k_1 and k_3 using least squares fitting.

Finally, the question remains: can we propose a solution for Ware Hall that will keep both the lab and faculty office comfortable? As a new option, we propose that temperature sensors be placed both in the lab and the faculty office. The weighted sum $T = \omega_1 F + \omega_2 L$ with $\omega_1 + \omega_2 = 1$ would then be used to control the HVAC. Determining the weights ω_1, ω_2 that manage to keep the lab and faculty office most comfortable forms yet another project for a differential equations class.

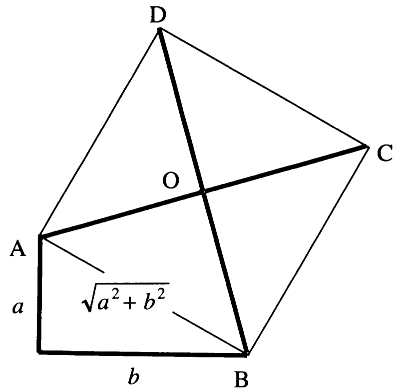
Acknowledgment. The authors thank Mr. Karl Reaves of the Mercer University Physical Plant for helpful conversations about HVAC systems.

References

1. P. Blanchard, R. L. Devaney, and G. R. Hall, *Differential Equations*, 2nd ed., Brooks/Cole, 2002.
2. CODEE (Consortium for Ordinary Differential Equations Experiments), *ODE Architect: The Ultimate ODE Power Tool*, Wiley, 1999.
3. P. S. Sansgiry and C. C. Edwards, A home heating model for calculus students, *College Mathematics Journal*, **27** (1996) 394–397.
4. H. J. Sauer, Jr. and R. H. Howell, *Principles of Heating, Ventilating and Air Conditioning*, American Society of Heating, Refrigerating and Air Conditioning Engineers, 1994.
5. J. Stewart, *Calculus: Concepts and Contexts*, 2nd ed., Brooks/Cole, 2001.

Construction Without Words

Here is another simple way to get the root mean square of a and b (see also Romero Márquez in the March 2001 issue, p. 118):



$$\mathrm{AO} = \mathrm{BO} = \mathrm{CO} = \mathrm{DO} = \sqrt{\frac{a^2+b^2}{2}}$$

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