Spectral Radius Algebras and Weighted Shifts of Finite Multiplicity

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Abstract

We consider the Spectral radius algebra associated with a weighted shift of finite multiplicity. When the weighted shift is injective, we describe the structure of this algebra. This leads to a necessary and sufficient condition for there to exist a nontrivial invariant subspace for the Spectral radius algebra. This result is then generalized to noninjective weighted shifts of finite multiplicity.

1 Introduction

Let \( H \) and \( K \) be Hilbert spaces over the field of complex numbers, \( \mathbb{C} \). Let \( \mathcal{L}(H, K) \) consist of all the bounded linear operators \( T : H \to K \). When \( K = H \), we will write \( \mathcal{L}(H) \) for \( \mathcal{L}(H, K) \). Let \( \{H_k\}_{k \in \mathbb{N}} \) be a sequence of complex Hilbert spaces such that \( \dim(H_k) = \dim(H_1) \), for all \( k \in \mathbb{N} \). Recall that \( H = \bigoplus_{k \in \mathbb{N}} H_k \) consists of sequences \( x = (x_1, x_2, \ldots) \) such that \( x_k \in H_k \), \( \sum_{k \in \mathbb{N}} \|x_k\|^2 < \infty \), and \( H \) is a Hilbert space with the inner product \( \langle x, y \rangle = \sum_{k \in \mathbb{N}} \langle x_k, y_k \rangle \).

Let \( \{A_k\}_{k \in \mathbb{N}} \) be a uniformly bounded sequence of operators \( A_k \in \mathcal{L}(H_k, H_{k+1}) \), a weighted shift \( W \) is defined by \( W(x_1, x_2, x_3, \ldots) = (0, A_1 x_1, A_2 x_2, \ldots) \). We will often write \( W \sim (A_n) \) to emphasize that \( W \) has a weight sequence of \( \{A_n\}_{n \in \mathbb{N}} \). We will assume that \( A_n \neq 0 \), for every \( n \in \mathbb{N} \), because if \( A_n \) were indeed zero, for some \( n \in \mathbb{N} \), then \( W \) could be written as a direct sum of two shifts. Since \( \dim(H_k) = \dim(H_1) \), for each \( k \in \mathbb{N} \), we will refer to \( \dim(H_1) \) as the multiplicity of \( W \). The commutant of \( W \), denoted by \( \{W\}' \), has already been studied in [4] under the assumption that all the weights are invertible. It was shown that \( \{W\}' \) is always a proper subset of the operators with a block lower triangular matrix relative to the decomposition \( H = \bigoplus_{k \in \mathbb{N}} H_k \). Hence, \( W \) always has many nontrivial hyperinvariant subspaces.

Recently, interest has risen in extending these results beyond the commutant to two other algebras associated with \( W \): the Deddens algebra \( \mathcal{D}_W \) and the Spectral radius algebra \( \mathcal{B}_W \). These algebras always satisfy the property \( \{W\}' \subseteq \mathcal{D}_W \subseteq \mathcal{B}_W \). When one of these containments is proper, the existence of a nontrivial invariant subspace (n.i.s.) for the larger algebra yields a stronger result than the existence of a hyperinvariant subspace. This gives us two important questions to investigate: when are the inclusions \( \{W\}' \subseteq \mathcal{D}_W \subseteq \mathcal{B}_W \) proper and when does there exists a n.i.s. for \( \mathcal{D}_W \) or \( \mathcal{B}_W \)? Weighted shifts of multiplicity one were already studied in [10]. It was shown that \( \{W\}' \) is always properly contained.
in $\mathcal{D}_W$ and that the weak closure of $\mathcal{D}_W$ is properly contained in $\mathcal{B}_W$ if and only if $W$ is neither bounded below nor quasinilpotent. Furthermore, it was shown that $\mathcal{D}_W$ has a n.i.s. if and only if $W$ is not bounded below, and $\mathcal{B}_W$ has a n.i.s. if and only if $W$ is quasinilpotent. In [11], the Deddens algebra associated with $W$ was studied in more detail in the case when $W$ has finite multiplicity. The space $\mathcal{D}_W(i,j) \subset \mathcal{L}(\mathcal{H}_j, \mathcal{H}_i)$ was found to be very instrumental in this process. Operators in this space can be viewed as matrices in $\mathcal{D}_W$ which are zero outside the $(i,j)$ block (a formal definition will come later). It was then shown that $\{W\}'$ is always properly contained in $\mathcal{D}_W$ and that $\mathcal{D}_W$ has a n.i.s. if and only if $W$ is not bounded below or $\mathcal{D}_W(1,1) \neq \mathcal{L}(\mathcal{H}_1)$.

This paper can be viewed as a sequel to [10] and [11] and, for our main results, we will continue to assume that $W$ has finite multiplicity. The first major result (Corollary 3.5) states that if $W$ is neither bounded below nor quasinilpotent, then the weak closure of $\mathcal{D}_W$ is a proper subalgebra of $\mathcal{B}_W$. However, the converse is no longer true when the multiplicity of $W$ is larger than one and we will give an example to illustrate this. The second main result (Theorem 4.6) is that if $W$ is injective, then $\mathcal{B}_W$ has a n.i.s. if and only if $W$ is quasinilpotent or $\mathcal{B}_W(1,1) \neq \mathcal{L}(\mathcal{H}_1)$. Here, $\mathcal{B}_W(i,j)$ is the Spectral radius algebra analogue of $\mathcal{D}_W(i,j)$.

We then extend this result to noninjective weighted shifts (Theorem 5.7). Namely, we show that there exists a n.i.s. for $\mathcal{B}_W$ if and only if $W$ is quasinilpotent or $\mathcal{B}_W(n,n) \neq \mathcal{L}(\mathcal{H}_n)$, for some $n \in \mathbb{N}$.

In Section 2, definitions of the Deddens and Spectral radius algebras are given. From here, the spaces $\mathcal{D}_W(i,j)$ and $\mathcal{B}_W(i,j)$ are formally introduced and we will review some of the basic facts about the Deddens and Spectral radius algebras. In Section 3, we demonstrate why the spaces $\mathcal{B}_W(i,j)$ are useful in the study of the Spectral radius algebra. This will lead to a partial answer to the question of when is the containment $\mathcal{D}_W \subset \mathcal{B}_W$ is proper. In Section 4, we begin to focus on injective weighted shifts of finite multiplicity. The existence of a n.i.s. for $\mathcal{B}_W$ is considered and necessary and sufficient conditions are found for $\mathcal{B}_W$ to have a n.i.s. Finally, in Section 5 we consider what happens if we do not assume that the weights of $W$ are invertible. This will lead to a generalization of the results established in Section 4 about the existence of a n.i.s.

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## 2 Preliminaries

In this section, we will define the Deddens and Spectral radius algebras as well as the spaces $\mathcal{D}_W(i,j)$ and $\mathcal{B}_W(i,j)$. We will then review some basic results about these spaces.

As in the introduction, let $\{\mathcal{H}_k\}_{k \in \mathbb{N}}$ be a sequence of complex Hilbert spaces of the same dimension and let $\mathcal{H} = \oplus_{k \in \mathbb{N}} \mathcal{H}_k$. If $A \in \mathcal{L}(\mathcal{H})$ is an invertible operator, then we can define
the set \( \{ T \in \mathcal{L}(\mathcal{H}) : \sup_{n \in \mathbb{N}} \| A^n T A^{-n} \| < \infty \} \). This set forms an algebra and was initially studied by Deddens in [3], so we call it the Deddens algebra associated to \( A \) and denote it by \( D_A \). Equivalently, it can be shown that \( T \in D_A \) if and only if there exists \( M > 0 \) such that

\[
\| A^n T x \| \leq M \| A^n x \|, \quad \text{for all} \ n \in \mathbb{N} \text{ and for all} \ x \in \mathcal{H}.
\]  

(1)

Since we are interested in studying weighted shifts, which are not invertible, we will take (1) as the definition of the Deddens algebra associated to \( A \). One can now quickly see that \( \{ A \}' \subset D_A \).

For an arbitrary operator \( A \in \mathcal{L}(\mathcal{H}) \), let \( r(A) \) denote the spectral radius of \( A \) and let \( d_m = \frac{m}{1 + m r(A)} \), where \( m \) is a positive integer. The operator \( R_m \) is defined to be the positive square root of \( \sum_{n=0}^{\infty} d_m^{2n} A^n A^n \). Note that this sum does converge in norm to a positive invertible operator. The Spectral radius algebra associated to \( A \) is defined to be \( B_A = \{ T \in \mathcal{L}(\mathcal{H}) : \sup_{m \in \mathbb{N}} \| R_m T R_m^{-1} x \| < \infty \} \). This algebra was first studied in [5] in the case that \( A \) is a compact operator. There, it was shown that there always exists a n.i.s. for \( B_A \) when \( A \) is compact. Since then, Spectral radius algebra have been further studied in [1, 2, 6–10]. Furthermore, one can show (cf. [7, Proposition 1]) that \( T \in B_A \) if and only if there exists \( M > 0 \) such that

\[
\sum_{n=0}^{\infty} d_m^{2n} \| A^n T x \|^2 \leq M \sum_{n=0}^{\infty} d_m^{2n} \| A^n x \|^2, \quad \text{for all} \ m \in \mathbb{N} \text{ and for all} \ x \in \mathcal{H}.
\]  

(2)

By (1) and (2), we can now confirm that \( \{ A \}' \subset D_A \subset B_A \).

In [11], the spaces \( D_W(i, j) \) were found to be very useful in the study of Deddens algebras associated to weighted shifts. Similarly, we will define \( B_W(i, j) \) and we will see how these vector spaces determine the weak closure of \( B_W \) in the next section. To each operator \( T \in \mathcal{L}(\mathcal{H}) \), we can associate a matrix \( \{ T_{ij} \}_{i,j \in \mathbb{N}} \) where \( T_{ij} \in \mathcal{L}(\mathcal{H}_j, \mathcal{H}_i) \). Meanwhile, for any \( l, k \in \mathbb{N} \) and for any \( A \in \mathcal{L}(\mathcal{H}_k, \mathcal{H}_l) \), we define \( \overline{A} \) to be the operator in \( \mathcal{L}(\mathcal{H}) \) whose matrix is given by

\[
\overline{A}_{ij} = \begin{cases} 
0 & \text{if} \ (i, j) \neq (l, k) \\
A & \text{if} \ (i, j) = (l, k)
\end{cases}
\]

(3)

Thus, \( \| \overline{A} \| = \| A \| \) and if \( B \in \mathcal{L}(\mathcal{H}_k, \mathcal{H}_l) \), then \( \overline{AB} = \overline{A} \overline{B} \). The space \( D_W(i, j) \) is then defined by

\[
D_W(i, j) = \{ T \in \mathcal{L}(\mathcal{H}_j, \mathcal{H}_i) : \overline{T} \in D_W \}
\]  

(4)

and \( B_W(i, j) \) is defined by

\[
B_W(i, j) = \{ T \in \mathcal{L}(\mathcal{H}_j, \mathcal{H}_i) : \overline{T} \in B_W \}.
\]  

(5)

Let \( T \in \mathcal{L}(\mathcal{H}_j, \mathcal{H}_i) \) and let \( x = (x_1, x_2, x_3, \ldots) \in \mathcal{H} \). Then, for every \( n \in \mathbb{N} \), we have that \( ||W^n T x|| = ||A_{i+n-1} A_{i+n-2} \cdots A_i T x_j|| \). As we are using the definition of \( B_W \) given in (2), we will often come across the sum

\[
\sum_{n=0}^{\infty} d_m^{2n} \| W^n T x \|^2 = \| T x_j \|^2 + \sum_{n=1}^{\infty} d_m^{2n} \| A_{i+n-1} A_{i+n-2} \cdots A_i T x_j \|^2.
\]
Instead of writing \( \|Tx_j\|_2^2 \) separately, we will say that

\[
\sum_{n=0}^{\infty} d_m^n \|W^n Tx\|_2^2 = \sum_{n=0}^{\infty} d_m^n \|A_{i+n-1}A_{i+n-2} \cdots A_i T x_j\|_2^2
\]

with the understanding that \( A_{i+n-1}A_{i+n-2} \cdots A_i \) is the identity on \( \mathcal{H}_i \), for \( n = 0 \).

3 General Weighted Shifts

In this section, we will explain why our interest in \( \mathcal{B}_W(i,j) \) is justified. We will also give sufficient conditions for the weak closure of \( \mathcal{D}_W \) to be a proper subset of \( \mathcal{B}_W \). For each operator \( W \), the existence of a n.i.s. for \( \mathcal{B}_W \) yields a strictly stronger result than the existence of a n.i.s. for \( \mathcal{D}_W \). We start off with a proposition which makes the calculation of \( \mathcal{B}_W(i,j) \) easier.

**Proposition 3.1** Let \( W \sim (A_n) \) be a weighted shift and let \( T \in \mathcal{L}(\mathcal{H}_j, \mathcal{H}_i) \). Then \( T \) belongs to \( \mathcal{B}_W(i,j) \) if and only if there exists \( M > 0 \) such that

\[
\sum_{n=0}^{\infty} d_m^n \|A_{i+n-1}A_{i+n-2} \cdots A_i T x\|_2^2 \leq M \sum_{n=0}^{\infty} d_m^n \|A_{j+n-1}A_{j+n-2} \cdots A_j x\|_2^2,
\]

for all \( x \in \mathcal{H}_j \) and for all \( m \in \mathbb{N} \).

**Proof:** Let \( x = (x_1, x_2, x_3, \ldots) \in \mathcal{H} \) such that \( x_k = 0 \), for all \( k \neq j \), and let \( T \in \mathcal{B}_W(i,j) \).

Since \( \|W^n Tx\| = \|A_{i+n-1}A_{i+n-2} \cdots A_i T x_j\| \) and \( \|W^n x\| = \|A_{j+n-1}A_{j+n-2} \cdots A_j x_j\| \), it follows from (2) that (6) holds.

In the other direction, let \( x = (x_1, x_2, x_3, \ldots) \in \mathcal{H} \), where \( x_k \in \mathcal{H}_k \), for all \( k \in \mathbb{N} \), and let \( T \in \mathcal{L}(\mathcal{H}_j, \mathcal{H}_i) \). Assume that there exists \( M > 0 \) such that

\[
\sum_{n=0}^{\infty} d_m^n \|A_{i+n-1}A_{i+n-2} \cdots A_i T x_j\|_2^2 \leq M \sum_{n=0}^{\infty} d_m^n \|A_{j+n-1}A_{j+n-2} \cdots A_j x_j\|_2^2,
\]

for all \( x_j \in \mathcal{H}_j \) and for all \( m \in \mathbb{N} \). Note that, for any \( n \in \mathbb{N} \), we have that

\[
\|W^n x\|_2 = \sum_{k=1}^{\infty} \|A_{k+n-1}A_{k+n-2} \cdots A_k x_k\|_2^2,
\]

because \( \{A_{k+n-1}A_{k+n-2} \cdots A_k x_k\}_{k \in \mathbb{N}} \) forms an orthogonal set in \( \mathcal{H} \). In particular, this implies that \( \|A_{k+n-1}A_{k+n-2} \cdots A_k x_k\| \leq \|W^n x\|_2 \), for all \( k, n \in \mathbb{N} \). Therefore,

\[
\sum_{n=0}^{\infty} d_m^n \|W^n T x\|_2^2 = \sum_{n=0}^{\infty} d_m^n \|A_{i+n-1}A_{i+n-2} \cdots A_i T x_j\|_2^2
\]

\[
\leq M \sum_{n=0}^{\infty} d_m^n \|A_{j+n-1}A_{j+n-2} \cdots A_j x_j\|_2^2 \leq M \sum_{n=0}^{\infty} d_m^n \|W^n x\|_2^2
\]
This proposition allows us to quickly find many operators in $B_W$. Namely, the identity operator on $H_k$, which we will denote by $I_k$, belongs to $B_W(k, k)$ and each weight of $W$, $A_k \in \mathcal{L}(H_k, H_{k+1})$, belongs to $B_W(k+1, k)$.

**Proposition 3.2** Let $W \sim (A_n)$ be a weighted shift and let $I_k$ be the identity on $H_k$. Then $I_k \in B_W(k, k)$ and $A_k \in B_W(k+1, k)$.

**Proof:** For the identity, equality in (6) holds with $M = 1$. When $T = A_k$, we have that
\[
\|A_{k+n}A_{k+n-1} \cdots A_{k+1}Tx\| \leq \|A_{k+n}\|\|A_{k+n-1}A_{k+n-2} \cdots A_kx\|,
\]
for all $n \geq 0$ and for all $x \in H_k$. Hence, (6) holds with $M = \|W\|^2$ and it follows that $A_k \in B_W(k+1, k)$, for all $k \in \mathbb{N}$. □

Note that $I_k$ is the projection operator in $L(H)$ whose range is the direct summand $H_k$. This means that if $T \in B_W$ and $T$ has a matrix $T_{ij}$ relative to the decomposition $H = \oplus_{k \in \mathbb{N}} H_k$, then $T_{ij} = \overline{T_{ij}} T_{ij} \in B_W$ and we obtain the following result.

**Corollary 3.3** Let $W$ be a weighted shift and let $T \in B_W$ have a matrix of $T_{ij}$ relative to the decomposition $H = \oplus_{k \in \mathbb{N}} H_k$. Then $T_{ij} \in B_W(i, j)$.

Due to Corollary 3.3, an operator $T \in B_W$ can be written as a weak limit of operators of the form
\[
T_n = \sum_{i,j=1}^{n} T_{ij}, \quad \text{where} \quad T_{ij} \in B_W(i, j).
\]

Furthermore, an operator $T$ belongs to the weak closure of $B_W$ if and only if $T_{ij}$ belongs to the weak closure of $B_W(i, j)$. If $W$ has finite multiplicity, then $B_W(i, j) \subset \mathcal{L}(H_j, H_i)$ is finite dimensional and therefore weakly closed. In this case, $T$ belongs to the weak closure of $B_W$ if and only if $T_{ij} \in B_W(i, j)$, for all $i, j \in \mathbb{N}$. In Section 4, we will more closely investigate the relationships among among the spaces $B_W(i, j)$ in order to study the weak closure of $B_W$.

Before we move onto weighted shifts of finite multiplicity, knowing when $A_k^{-1} \in B_W$ will play an important role. In fact, we will show in the next section that if $W$ is an injective weighted shift of finite multiplicity and $A_k^{-1} \in B_W(k, k+1)$, then the weak closure of $B_W$ is completely determined by $B_W(1, 1)$.

**Theorem 3.4** Let $W \sim (A_n)$ be a weighted shift such that $A_k^{-1}$ exists, for some $k \in \mathbb{N}$. Then $A_k^{-1} \in B_W(k, k+1)$ if and only if $r(W) > 0$. 

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PROOF: Let $A_k^{-1} \in \mathcal{B}_W(k, k + 1)$, for some $k \in \mathbb{N}$. Proposition 3.1 implies that there exists $M > 0$ such that
\[
\sum_{n=0}^{\infty} d_m^{2n} \|A_{k+n-1}A_{k+n-2} \cdots A_k A^{-1}_k \| x \|^2 \leq M \sum_{n=0}^{\infty} d_m^{2n} \|A_{k+n}A_{k+n-1} \cdots A_{k+1}x \|^2, \quad (7)
\]
for all $x \in \mathcal{H}_{k+1}$ and for all $m \in \mathbb{N}$. The left hand side of this inequality can be rewritten as
\[
\sum_{n=0}^{\infty} d_m^{2n} \|A_{k+n-1}A_{k+n-2} \cdots A_k A^{-1}_k \| x \|^2 = \|A^{-1}_k x \|^2 + d_m^2 \sum_{n=0}^{\infty} d_m^{2n} \|A_{k+n}A_{k+n-1} \cdots A_{k+1}x \|^2. \quad (8)
\]
Using (7) and (8), we can determine that the inequality
\[
d_m^2 \sum_{n=0}^{\infty} d_m^{2n} \|A_{k+n}A_{k+n-1} \cdots A_{k+1}x \|^2 \leq M \sum_{n=0}^{\infty} d_m^{2n} \|A_{k+n}A_{k+n-1} \cdots A_{k+1}x \|^2 \quad (9)
\]
holds, for all $m \in \mathbb{N}$ and for all $x \in \mathcal{H}_{k+1}$. This implies that $\{d_m\}_{m \in \mathbb{N}}$ must be a bounded sequence which only happens when $r(W) > 0$.

To complete the proof, we will assume that $r(W) > 0$ and $A_k^{-1}$ exists, for some $k \in \mathbb{N}$, and then show that $A_k^{-1} \in \mathcal{B}_W(k, k + 1)$. If $r(W) > 0$, then $d_m \leq \frac{1}{r(W)}$, for all $m \in \mathbb{N}$, and (8) implies that
\[
\sum_{n=0}^{\infty} d_m^{2n} \|A_{k+n-1}A_{k+n-2} \cdots A_k A^{-1}_k \| x \|^2 \\
\leq \|A^{-1}_k \|^2 \|x\|^2 + \frac{1}{r(W)^2} \sum_{n=0}^{\infty} d_m^{2n} \|A_{k+n}A_{k+n-1} \cdots A_{k+1}x \|^2 \\
\leq \|A^{-1}_k \|^2 \sum_{n=0}^{\infty} d_m^{2n} \|A_{k+n}A_{k+n-1} \cdots A_{k+1}x \|^2 + \frac{1}{r(W)^2} \sum_{n=0}^{\infty} d_m^{2n} \|A_{k+n}A_{k+n-1} \cdots A_{k+1}x \|^2 \\
= \left( \|A^{-1}_k \|^2 + \frac{1}{r(W)^2} \right) \sum_{n=0}^{\infty} d_m^{2n} \|A_{k+n}A_{k+n-1} \cdots A_{k+1}x \|^2.
\]
Applying Proposition 3.1, we have proven that $A_k^{-1} \in \mathcal{B}_W(k, k + 1)$ because (6) holds when $M = \|A^{-1}_k \|^2 + \frac{1}{r(W)^2}$.

We can finally give a class of operators for which the weak closure of $\mathcal{D}_W$ is properly contained in $\mathcal{B}_W$. It was shown in [11] that if $W$ is a weighted shift whose weights are all invertible, then $\overline{A_n^{-1}}$ belongs to $\mathcal{D}_W$, for all $n \in \mathbb{N}$, if and only if $W$ is bounded below. It was also shown that Corollary 3.3 can be applied to $\mathcal{D}_W$, so if $W$ has finite multiplicity and $T \in \mathcal{L}(\mathcal{H}_j, \mathcal{H}_i)$, then $T$ belongs to the weak closure of $\mathcal{D}_W$ if and only if $T \in \mathcal{D}_W(i, j)$. In other words, if $W \sim (A_n)$ is an injective weighted shift of finite multiplicity that is not bounded below, then $\overline{A_n^{-1}}$ does not belong to the weak closure of $\mathcal{D}_W$, for any $n \in \mathbb{N}$. However, we have just shown that $\overline{A_n^{-1}} \in \mathcal{B}_W$ when $W$ is not quasinilpotent so we have the following corollary.
Corollary 3.5 Let $W$ be an injective weighted shift of finite multiplicity such that $W$ is neither quasinilpotent nor bounded below. Then the weak closure of $D_W$ is properly contained in $B_W$.

Examples of weighted shifts which satisfy the assumptions of Corollary 3.5 are easy to come by. For example, let $\{B_n\}_{n \in \mathbb{N}}$ be a sequence of invertible operators such that the sequence of their lower bounds is not bounded below, and let $\{U_n\}_{n \in \mathbb{N}}$ be a sequence of unitary operators. Let $A_n = B_k$, if $n = 2^k$, and let $A_n = U_n$ otherwise. Then $W \sim (A_n)$ is not bounded below. However, due to arbitrarily long sequences of unitary operators, we have that $\|W^n\| \geq 1$, for all $n \in \mathbb{N}$. Hence, the spectral radius formula tells us that $r(W) \geq 1$. Therefore, there are many weighted shifts for which the weak closure of $D_W$ is properly contained in $B_W$.

One may be curious if Corollary 3.5 could also be applied to weighted shifts of infinite multiplicity. However, this is not the case. In particular, it is possible to construct an example of a weighted shift $W$, with invertible weights, which is neither bounded below nor quasinilpotent, yet the weak closures of $D_W$ and $B_W$ coincide. This is because $A_n^{-1}$ may now belong to the weak closure of $D_W(n, n+1)$ so $A_n^{-1}$ no longer provides us with an example of an operator outside the weak closure of $D_W$. In terms of invariant subspaces, this weighted shift has the property that $M$ is a n.i.s. for $D_W$ if and only if $M$ is a n.i.s. for $B_W$.

The converse of Corollary 3.5 is true if $W$ has multiplicity one (see [10]). However, we will demonstrate in the next example that there exists an injective weighted shift $W$ of finite multiplicity which is bounded below but the weak closure of $D_W$ is a proper subalgebra of $B_W$. In Section 5, we will extend Corollary 3.5 to weighted shifts which are not injective thus giving us many more weighted shifts for which $B_W$ is larger than $D_W$.

Example Let dim($\mathcal{H}_n$) = 2, for all $n \in \mathbb{N}$. Let

$$A = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix},$$

and define $W$ to be the weighted shift with weight sequence


It is clear from this definition that $W$ is an injective weighted shift which is bounded below. Also, one can compute that $\|W^n\| = 2^n$, for all $n \in \mathbb{N}$, whence $r(W) = 2$.

It was shown in [11] that

$$D_W(1, 1) = \left\{ \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} \in \mathcal{L}(\mathcal{H}_1) : a, b \in \mathbb{C} \right\}.$$ 

We will show that

$$T = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \in B_W(1, 1)$$
which implies that the containment $D_W(1,1) \subset B_W(1,1)$ is proper and that the weak closure of $D_W$ is a proper subalgebra of $B_W$.

By Proposition 3.1, we know that $T \in B_W(1,1)$ if and only if there exists $M > 0$ such that
\[ \sum_{n=0}^{\infty} d^2_n \| A_n \cdots A_1 T x \|^2 \leq M \sum_{n=0}^{\infty} d^2_n \| A_n \cdots A_1 x \|^2, \]
for all $n \in \mathbb{N}$ and for all $x \in H_1$. Let $e_1 = (1, 0)^T$, $e_2 = (0, 1)^T$, and let $x = x_1 e_1 + x_2 e_2$. Then
\[ \sum_{n=0}^{\infty} d^2_n \| A_n \cdots A_1 T x \|^2 = |x_1|^2 \sum_{n=0}^{\infty} d^2_n \| A_n \cdots A_1 e_1 \|^2 + |x_2|^2 \sum_{n=0}^{\infty} d^2_n \| A_n \cdots A_1 e_2 \|^2. \]
and
\[ \sum_{n=0}^{\infty} d^2_n \| A_n \cdots A_1 x \|^2 = |x_1|^2 \sum_{n=0}^{\infty} d^2_n \| A_n \cdots A_1 e_1 \|^2 + |x_2|^2 \sum_{n=0}^{\infty} d^2_n \| A_n \cdots A_1 e_2 \|^2. \]
Thus, (10) will hold if we show that there exists $M > 0$ such that
\[ \sum_{n=0}^{\infty} d^2_n \| A_n \cdots A_1 e_1 \|^2 \leq M \sum_{n=0}^{\infty} d^2_n \| A_n \cdots A_1 e_2 \|^2, \]
for all $m \in \mathbb{N}$. We will actually prove that there exists $M > 0$ such that
\[ \sum_{n=0}^{\infty} d^2_n \| A_n \cdots A_1 e_1 \|^2 \leq M, \]
for all $m \in \mathbb{N}$ and (11) will follow because
\[ \sum_{n=0}^{\infty} d^2_n \| A_n \cdots A_1 e_2 \|^2 \geq 1, \]
for all $m \in \mathbb{N}$.

Let $k \geq 1$ and let $(2k-1)k = 1 + 2 + \cdots + (2k-1) < n \leq 1 + 2 + \cdots + (2k) = (2k+1)k$. Then
\[ \| A_n A_{n-1} \cdots A_1 e_1 \| = 2^{1+3+\cdots+(2k-1)} = 2^{(k-1)^2}. \]
Meanwhile, if $(2k+1)k = 1 + 2 + \cdots + (2k) < n \leq 1 + 2 + \cdots + (2k+1) = (2k+1)(k+1)$, then
\[ \| A_n A_{n-1} \cdots A_1 e_1 \| = 2^{1+3+\cdots+(2k-1)} \cdot 2^{n-(1+2+\cdots+(2k))} = 2^{(k-1)^2+n-k(2k+1)}. \]
Therefore, if $(2k-1)k < n \leq k(2k+1)$, then (12) implies that
\[ \| A_n A_{n-1} \cdots A_1 e_1 \|^{\frac{2}{n}} = 2^{2(k-1)^2/n} \leq 2^{2(k-1)^2/|2k-1)k|}, \]
and if $k(2k+1) < n \leq (2k+1)(k+1)$, then (13) implies that
\[ \| A_n A_{n-1} \cdots A_1 e_1 \|^{\frac{2}{n}} = 2^{2((k-1)^2+n-k(2k+1))/n} \leq 2^{2(k-1)^2+(2k+1)(k+1)-k(2k+1))|/|2k-1)k|}. \]
One can quickly confirm that
\[
\lim_{k \to \infty} 2^{(k-1)^2}/[(2k-1)k] = \lim_{k \to \infty} 2^{(k-1)^2+(2k+1)(k+1)-(k(2k+1))}/[(2k-1)k] = 2,
\]
so it follows that \(\limsup_{n \to \infty} \|A_nA_{n-1} \cdots A_1 e_1\|^2 \leq 2\). This implies that the function
\[
f(z) = \sum_{n=0}^{\infty} \|A_nA_{n-1} \cdots A_1 e_1\|^2 z^n
\]
has a radius of convergence of \(R \geq \frac{1}{2}\). Hence, there exists \(M > 0\) such that \(f(z) \leq M\), for all \(z \in \{z \in \mathbb{C} : |z| \leq \frac{1}{4}\}\). Since \(d_n^2 = \left(\frac{m}{1+2m}\right)^2 \leq \frac{1}{4}\), for all \(m \in \mathbb{N}\), we can now conclude that
\[
\sum_{n=0}^{\infty} d_n^{2m} \|A_nA_{n-1} \cdots A_1 e_1\| \leq M,
\]
for all \(m \in \mathbb{N}\). Therefore, \(T \in B_W(1, 1)\).

In a similar manner to how we proved that \(T \in B_W(1, 1)\), it can also be shown that
\[
\begin{bmatrix}
0 & 0 \\
1 & 0
\end{bmatrix} \in B_W(1, 1).
\]
This combined with the fact that the diagonal matrices are contained in \(D_W(1, 1) \subset B_W(1, 1)\) would imply that \(B_W(1, 1) = L(H_1)\). In the next section, we will show that this equality, along with the fact that \(r(W) > 0\), implies that \(B_W\) is weakly dense in \(L(H)\).

### 4 Injective Weighted Shifts of Finite Multiplicity

For the remainder of this paper, we will assume that \(W\) has finite multiplicity so that we can determine more about the structure of the operators in \(B_W\). For injective weighted shifts, we will describe a relationship between \(B_W(i, j)\) and \(B_W(k, l)\). However, this relation disappears when \(W\) is not injective and we will discuss what can be said about such shifts in Section 5. The main result of this section is Theorem 4.6 which is the statement of the necessary and sufficient conditions for when there exists a n.i.s. for \(B_W\).

If \(W\) is an injective weighted shift of finite multiplicity, then \(A_k\) is an invertible operator, for each \(k \in \mathbb{N}\). The results in this section take advantage of this fact and demonstrate that if we know which operators belong to \(B_W(i, j)\), then we can use this information to help determine the operators in \(B_W(i+1, j)\), \(B_W(i, j-1)\), and \(B_W(i+1, j+1)\).

**Lemma 4.1** Let \(W \sim (A_n)\) be an injective weighted shift of finite multiplicity and let \(i\) and \(j\) be positive integers. Then multiplication on the left by \(A_i\) is an injective linear transformation from \(B_W(i, j)\) into \(B_W(i+1, j)\) and multiplication on the right by \(A_j\) is an injective linear transformation from \(B_W(i, j+1)\) into \(B_W(i, j)\).

**Proof:** Let \(T \in B_W(i, j)\). Then \(A_i \overline{T} \in B_W\) by Corollary 3.2 and \(A_i T : H_j \to H_{i+1}\). Hence, multiplication by \(A_i\) on the left is a well-defined linear transformation from \(B_W(i, j)\).
into $B_W(i, j)$). Furthermore, this mapping is injective because $A_i$ is an invertible operator. Similarly, multiplication by $A_j$ on the right is an injective linear transformation from $B_W(i, j + 1)$ into $B_W(i, j)$. 

The previous lemma comes as no surprise as it is an application of the fact that $B_W$ is closed under multiplication. The next lemma will show that the function corresponding to the combined multiplication of $A_i$ on the left and $A_j^{-1}$ on the right is an isomorphism from $B_W(i, j)$ onto $B_W(i + 1, j + 1)$. This is very surprising because $A_j^{-1}$ does not necessarily belong to $B_W$ so there is no apparent reason why this function would be well-defined.

**Lemma 4.2** Let $W \sim (A_n)$ be an injective weighted shift of finite multiplicity and let $i$ and $j$ be positive integers. Then the mapping $\varphi : B_W(i, j) \to B_W(i + 1, j + 1)$ defined by $\varphi(T) = A_iTA_j^{-1}$ is an isomorphism of vector spaces.

**Proof:** Let $T \in B_W(i, j)$. Since $W$ is an injective weighted shift of finite multiplicity, $A_j^{-1}$ exists, for all $n \in \mathbb{N}$, and it is not hard to see that the mapping $T \mapsto A_iTA_j^{-1}$ is an injective linear transformation from $B_W(i, j)$ into $\mathcal{L}(H_{j+1}, H_{i+1})$. We will now show that $A_iTA_j^{-1} \in B_W(i + 1, j + 1)$. Since $T \in B_W(i, j)$, Proposition 3.1 tells us that there exists $M > 0$ such that

$$\sum_{n=0}^{\infty} d_m^2 \|A_{i+n-1}A_{i+n-2} \cdots A_iTx\|^2 \leq M \sum_{n=0}^{\infty} d_m^2 \|A_{j+n-1}A_{j+n-2} \cdots A_jx\|^2,$$

for all $x \in H_j$ and for all $n \in \mathbb{N}$. Let $y \in H_{j+1}$. Note that $d_m$ is a monotone increasing function so that $d_m^{-2} \leq d_1^{-2}$, for all $m \in \mathbb{N}$, and one can now show that

$$\sum_{n=0}^{\infty} d_m^2 \|A_{i+n}A_{i+n-1} \cdots A_i(A_iTA_j^{-1})y\|^2 = \frac{1}{d_m^2} \sum_{n=0}^{\infty} d_m^2(n+1) \|A_{i+n}A_{i+n-1} \cdots A_iTA_j^{-1}y\|^2$$

$$= \frac{1}{d_m^2} \sum_{n=1}^{\infty} d_m^2 \|A_{i+n}A_{i+n-2} \cdots A_iTA_j^{-1}y\|^2 \leq M \frac{1}{d_m^2} \sum_{n=0}^{\infty} d_m^2 \|A_{j+n-1}A_{j+n-2} \cdots A_jA_j^{-1}y\|^2$$

$$= M \left( \frac{\|A_j^{-1}y\|^2}{d_1^2} + \sum_{n=1}^{\infty} d_m^2(n-1) \|A_{j+n-1}A_{j+n-2} \cdots A_jy\|^2 \right)$$

$$\leq M \left( \frac{\|A_j^{-1}y\|^2}{d_1^2} + \sum_{n=0}^{\infty} d_m^2 \|A_{j+n}A_{j+n-1} \cdots A_jy\|^2 \right)$$

$$\leq M \left( \frac{\|A_j^{-1}y\|^2}{d_1^2} + 1 \sum_{n=0}^{\infty} d_m^2 \|A_{j+n}A_{j+n-1} \cdots A_jy\|^2 \right),$$

for all $m \in \mathbb{N}$ and for all $y \in H_{j+1}$. Therefore, $A_iTA_j^{-1} \in B_W$, by Proposition 3.1, and we can conclude that $\varphi : B_W(i, j) \to B_W(i + 1, j + 1)$ is well defined.
We will now show that the range of \( \varphi \) is \( B_W(i + 1, j + 1) \). Let \( X \in B_W(i + 1, j + 1) \). Since \( \varphi(A_i^{-1}XA_j) = X \), it suffices to prove that \( A_i^{-1}XA_j \in B_W(i, j) \). By Proposition 3.1, there exists \( M > 0 \) such that
\[
\sum_{n=0}^{\infty} d_m^{2n} \| A_{i+n}A_{i+n-1} \cdots A_{i+1}Xx \|^2 \leq M \sum_{n=0}^{\infty} d_m^{2n} \| A_{j+n}A_{j+n-1} \cdots A_{j+1}x \|^2,
\]
for all \( x \in H_{j+1} \) and for all \( m \in \mathbb{N} \). Let \( y \in H_j \). Then
\[
\sum_{n=0}^{\infty} d_m^{2n} \| A_{i+n-1}A_{i+n-2} \cdots A_i(A_i^{-1}XA_j)y \|^2
\]
\[
= \| A_i^{-1}XA_jy \|^2 + \sum_{n=1}^{\infty} d_m^{2n} \| A_{i+n-1}A_{i+n-2} \cdots A_{i+1}XA_jy \|^2
\]
\[
= \| A_i^{-1}XA_jy \|^2 + d_m^{2} \sum_{n=0}^{\infty} d_m^{2n} \| A_{i+n}A_{i+n-1} \cdots A_{i+1}XA_jy \|^2
\]
\[
\leq \| A_i^{-1}XA_jy \|^2 \| y \|^2 + Md_m^{-2} \sum_{n=0}^{\infty} d_m^{2n} \| A_{j+n}A_{j+n-1} \cdots A_{j+1}A_jy \|^2
\]
\[
\leq \| A_i^{-1}XA_jy \|^2 \| y \|^2 + Md_m^{-2} \sum_{n=0}^{\infty} d_m^{2n} \| A_{j+n}A_{j+n-1} \cdots A_{j+1}A_jy \|^2
\]
\[
\leq \| A_i^{-1}XA_jy \|^2 \| y \|^2 + Md_m^{-2} \sum_{n=0}^{\infty} d_m^{2n} \| A_{j+n} \cdots A_{j+1}A_jy \|^2
\]
for all \( m \in \mathbb{N} \) and for all \( y \in H_j \). The result now follows from Proposition 3.1. \( \square \)

We will now reformulate Lemmas 4.1 and 4.2 in terms of a relationship between the dimensions of various \( B_W(i, j) \).

**Corollary 4.3** Let \( W \) be an injective weighted shift of finite multiplicity. Then

1. \( \dim(B_W(i, j)) \leq \dim(B_W(l, k)) \), for all \( k \leq j \) and for all \( i \leq l \).

2. \( \dim(B_W(1, j)) = \dim(B_W(1 + n, j + n)) \), for all \( j \in \mathbb{N} \) and \( n \geq 0 \).

3. \( \dim(B_W(i, 1)) = \dim(B_W(i + n, 1 + n)) \) for all \( i \in \mathbb{N} \) and \( n \geq 0 \).

These results tell us that if we can determine \( B_W(1, j) \) and \( B_W(i, 1) \), then we know the weak closure of \( B_W \). The next figure helps give a picture of what is happening.
As we move from the \((i, j)\) block in the direction of the solid arrows, the dimension of \(B_W(i, j)\) cannot decrease. While moving in the direction of dashed arrows, the dimension of \(B_W(i, j)\) is constant.

The homomorphisms in Lemma 4.1 are invertible precisely when \(A_k^{-1} \in B_W(k, k + 1)\), for all \(k \in \mathbb{N}\). In this case, we have equality among dimensions and the weak closure of \(B_W\) is determined by \(B_W(1, 1)\). The following lemma formalizes this.

**Lemma 4.4** Let \(W \sim (A_n)\) be an injective weighted shift of finite multiplicity. Then, \(\dim(B_W(i, j)) = \dim(B_W(1, 1))\), for all \(i, j \in \mathbb{N}\), if and only if \(r(W) > 0\).

**Proof:** Assume that \(\dim(B_W(i, j)) = \dim(B_W(1, 1))\), for all \(i, j \in \mathbb{N}\). In particular, we have that \(\dim(B_W(1, 1)) = \dim(B_W(1, 2))\). By Lemma 4.1, multiplication by \(A_1\) on the right provides an injective linear transformation from \(B_W(1, 2)\) into \(B_W(1, 1)\). Since \(\dim(B_W(1, 2)) = \dim(B_W(1, 1)) < \infty\), this multiplication is an isomorphism of vector spaces.

By Proposition 3.2, we have that \(I_1 \in B_W(1, 1)\) implying that there exists \(X \in B_W(1, 2)\) such that \(XA_1 = I_1\). Therefore \(X = A_1^{-1} \in B_W(1, 2)\) and \(r(W) > 0\) by Theorem 3.4. If \(r(W) > 0\), then Theorem 3.4 implies that \(A_k^{-1} \in B_W\), for all \(k \in \mathbb{N}\). Thus multiplication by \(A_i^{-1}\) on the left is a linear transformation from \(B_W(i + 1, j)\) into \(B_W(i, j)\) and is the inverse of multiplication by \(A_i\) on the left. Similarly, multiplication by \(A_j^{-1}\) on the right is a linear transformation from \(B_W(i, j)\) into \(B_W(i, j + 1)\) and is the inverse of multiplication by \(A_j\) on the right. Therefore \(B_W(i, j)\) is isomorphic to \(B_W(i + 1, j)\) and \(B_W(i, j + 1)\), for all \(i, j \in \mathbb{N}\). This implies that \(B_W(i, j)\) is isomorphic to \(B_W(1, 1)\), for all \(i, j \in \mathbb{N}\). \(\Box\)

In particular, this lemma tells us that if \(r(W) = 0\), then the dimensions of \(B_W(i, j)\) cannot all be equal. The following result shows at least one place where the inequality is strict.

**Corollary 4.5** Let \(W \sim (A_n)\) be an injective, quasinilpotent weighted shift of finite multiplicity. Then \(\dim(B_W(1, 2)) < \dim(B_W(1, 1))\) and \(B_W(1, k)\) does not contain an invertible operator, for any \(k \geq 2\).
Proof: Let $W \sim (A_n)$ be an injective quasinilpotent weighted shift of finite multiplicity. By Corollary 4.3, we know that $\dim(B_W(1, 2)) \leq \dim(B_W(1, 1))$. Assume for the purpose of proof by contradiction that $\dim(B_W(1, 2)) = \dim(B_W(1, 1))$. Then multiplication by $A_1$ on the right is a vector space isomorphism from $B_W(1, 2)$ onto $B_W(1, 1)$. By Proposition 3.2, we know that $I_1 \in B_W(1, 1)$, so there exists $X \in B_W(1, 2)$ such that $XA_1 = I_1$. However, this implies that $X = A_1^{-1} \in B_W(1, 2)$ and, by Theorem 3.4, $W$ not quasinilpotent. Hence, $\dim(B_W(1, 2)) < \dim(B_W(1, 1))$.

Now we will show that $B_W(1, k)$ does not contain an invertible operator, for any $k \geq 2$. Once again, assume for the purpose of proof by contradiction that $X \in B_W(1, k)$ is an invertible operator, for some $k \geq 2$. Then multiplication by $X$ on the right yields an injective linear transformation from $B_W(1, 1)$ into $B_W(1, k)$. By Corollary 4.3, this would imply that $\dim(B_W(1, 1)) \leq \dim(B_W(1, k)) \leq \dim(B_W(1, 2))$. However, we know that this is not the case since we just proved that $\dim(B_W(1, 1)) > \dim(B_W(1, 2))$.

We now reach the main result in this section. We will characterize the weighted shifts for which $B_W$ is weakly dense in $\mathcal{L}(\mathcal{H})$. Then, for operators without this property, we will construct a n.i.s. for $B_W$.

**Theorem 4.6** Let $W \sim (A_n)$ be an injective weighted shift of finite multiplicity. The following are equivalent:

1. $r(W) > 0$ and $B_W(1, 1) = \mathcal{L}(\mathcal{H}_1)$.
2. The weak closure of $B_W$ is $\mathcal{L}(\mathcal{H})$.
3. There does not exist a n.i.s. for $B_W$.

Proof: We start by showing that the first statement implies the second. Let $W$ be a weighted shift which is not quasinilpotent such that $B_W(1, 1) = \mathcal{L}(\mathcal{H}_1)$. By Lemma 4.4, it follows that $B_W(i, j) = \mathcal{L}(\mathcal{H}_j, \mathcal{H}_i)$, for all $i, j \in \mathbb{N}$. If $T \in \mathcal{L}(\mathcal{H})$, then $T$ has a matrix $(T_{ij})_{i,j \in \mathbb{N}}$ with $T_{ij} \in \mathcal{L}(\mathcal{H}_j, \mathcal{H}_i) = B_W(i, j)$. Therefore, $T$ is a weak limit of $\sum_{i,j=1}^n T_{ij} \in B_W$ implying that $T$ is in the weak closure of $B_W$.

The fact that the second statement implies the third is obvious, so it remains to show that the third statement implies the first. We will actually prove the contrapositive, “if $r(W) = 0$ or $B_W(1, 1) \neq \mathcal{L}(\mathcal{H}_1)$ then there exists a n.i.s. for $B_W$.”

First, we will show that if $B_W(1, 1) = \mathcal{L}(\mathcal{H}_1)$ and $r(W) = 0$, then $B_W(1, j) = 0$, for all $j > 1$. By Corollary 4.3, this will imply that $B_W$ is block lower triangular and therefore has a n.i.s. Since $B_W(1, 1) = \mathcal{L}(\mathcal{H}_1)$, we know that $B_W(i, j) = \mathcal{L}(\mathcal{H}_j, \mathcal{H}_i)$, for all $i \geq j$, by Lemmas 4.1 and 4.2. Assume to the contrary that $B_W(1, j_0) \neq 0$, for some $1 < j_0 \in \mathbb{N}$. Let $A \in \mathcal{L}(\mathcal{H}_1) = B_W(1, 1)$, let $T \in \mathcal{L}(\mathcal{H}_2, \mathcal{H}_{j_0}) = B_W(j_0, 2)$, and let $0 \neq X \in B_W(1, j_0)$. Since $B_W$ is closed under multiplication, it follows that $AXT \in B_W$ with $AXT : \mathcal{H}_2 \to \mathcal{H}_1$. In other words, we have that $AXT \in B_W(1, 2)$. Since $A$ and $T$ were arbitrary, one can
quickly show that $B_W(1,2)$ contains every rank one operator in $\mathcal{L}(\mathcal{H}_2, \mathcal{H}_1)$. It follows that $B_W(1,2) = \mathcal{L}(\mathcal{H}_2, \mathcal{H}_1)$ and, in particular, $A_1^{-1} \in B_W(1,2)$. By Theorem 3.4, we can conclude that $r(W) > 0$. This proves that if $B_W(1,1) = \mathcal{L}(\mathcal{H}_1)$ and $r(W) = 0$, then $B_W$ is block lower triangular and it follows that there are many nontrivial invariant subspaces for $B_W$.

We can now complete the proof by considering the case when $B_W(1,1) \neq \mathcal{L}(\mathcal{H}_1)$ and constructing a n.i.s. for $B_W$. Since $B_W(1,1)$ is a finite dimensional subalgebra of $\mathcal{L}(\mathcal{H}_1)$, Burnside’s Theorem states that there exists a n.i.s. $M_1 \subset \mathcal{H}_1$ for $B_W(1,1)$. Define $M_k$ to be the subspace of $\mathcal{H}_k$ generated by \{Ax : x \in M_1, A \in B_W(k,1)\} and define $M$ to be the closure of $\oplus_{k \in \mathbb{N}} M_k$. We will show that $M$ is a n.i.s. for $B_W$. It suffices to show that $M$ is invariant for $T_{ij}$ whenever $T_{ij} \in B_W(i,j)$, for all $i, j \in \mathbb{N}$. Indeed if $T \in B_W$, then $T$ is a weak limit of $\sum_{r=1}^{n} T_{ij}$ and if $M$ is invariant for each $T_{ij}$, then $M$ is invariant for the sum $\sum_{r=1}^{n} T_{ij}$ and also for $T$. Furthermore, $M$ is nontrivial because we assumed that $M_1$ is nontrivial.

Thus, it remains to show that if $T \in B_W(i,j)$, then $\overline{T(M)} \subset M$. Let $T \in B_W(i,j)$ and let $y = (y_1, y_2, y_3, \ldots) \in M$ where $y_k \in M_k$, for all $k \in \mathbb{N}$. Then $Ty = (z_1, z_2, z_3, \ldots)$ where $z_k = 0$ for $k \neq i$ and $z_i = Ty_j$. From the definition of $M_j$, there exists $m \in \mathbb{N}$, $B_k \in B_W(j,1)$, and $x_k \in M_1$, $1 \leq k \leq m$, such that $y_j = \sum_{k=1}^{m} B_k x_k$. We can now conclude that $Ty_j = \sum_{k=1}^{m} TB_k x_k \in M_i$ because $TB_k x_k \in M_i$, for each $k$, $1 \leq k \leq m$. Therefore, $Ty \in M$, for every $y \in M$, and $M$ is a n.i.s. for $B_W$. □

5 Noninjective Weighted Shifts

When $W$ is not injective, Lemmas 4.1 and 4.2 do not hold and knowing $B_W(i,j)$ yields very little information about $B_W(i+1,j), B_W(i,j-1), \text{or } B_W(i+1,j+1)$. Essentially, we must check each value of $i$ and $j$ separately. Once again, the main goal is to discuss the existence of a n.i.s. for $B_W$. We will break up this section into two parts: the first on noninjective weighted shifts with a positive spectral radius and the second on noninjective, quasinilpotent weighted shifts.

5.1 Noninjective weighted shifts with a positive spectral radius

In this section, we will show that Theorem 4.6 does not hold for noninjective weighted shifts with positive spectral radiii. Namely, we will give an example of a weighted shift such that $B_W$ has a n.i.s. but $r(W) > 0$ and $B_W(1,1) = \mathcal{L}(\mathcal{H}_1)$. Before we construct this example, we will establish some facts about noninjective weighted shifts.

Lemma 5.1 Let $W \sim (A_n)$ be a noninjective weighted shift with $r(W) > 0$. Let $i \in \mathbb{N}$ and let $y \in \mathcal{H}_i$ such that $A_{i+k-1} A_{i+k-2} \cdots A_i y = 0$, for some $k \in \mathbb{N}$. Then $y \otimes x \in B_W(i,j)$, for all $j \in \mathbb{N}$ and for all $x \in \mathcal{H}_j$. 

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Proof: Let $y \in \mathcal{H}_i$ satisfy $A_{i+k-1}A_{i+k-2}\cdots A_i y = 0$, for some $k \in \mathbb{N}$, and let $x$ be an arbitrary vector in $\mathcal{H}_j$. Define $M$ to be
\[ M = ||x||^2 \sum_{n=0}^{k-1} \frac{1}{r(W)^{2n}} ||A_{i+n-1}A_{i+n-2}\cdots A_i y||^2. \]

Let $z \in \mathcal{H}_j$. Note that $d_m \leq \frac{1}{r(W)}$ when $r(W) > 0$, whence
\[
\sum_{n=0}^{\infty} d_m^{2n} ||A_{i+n-1}A_{i+n-2}\cdots A_i (y \otimes x) z||^2 = \sum_{n=0}^{k-1} d_m^{2n} ||A_{i+n-1}A_{i+n-2}\cdots A_i y||^2 |\langle z, x \rangle|^2 \leq M \cdot ||z||^2
\]
\[
\leq M \sum_{n=0}^{\infty} d_m^{2n} ||A_{j+n-1}A_{j+n-2}\cdots A_j z||^2,
\]
for all $m \in \mathbb{N}$ and for all $z \in \mathcal{H}_j$. By Proposition 3.1, it follows that $x \otimes y \in \mathcal{B}_W(i, j)$. □

This result allows us to extend Corollary 3.5 to a larger class of weighted shifts.

Corollary 5.2 Let $W$ be a noninjective weighted shift of finite multiplicity. If $W$ is not quasinilpotent and $\ker(W^n) \neq \ker(W)$, for some $n \in \mathbb{N}$, then the weak closure of $D_W$ is properly contained in $\mathcal{B}_W$.

Proof: Since $W$ is not injective, there exists $j \in \mathbb{N}$ and a nonzero vector $x \in \mathcal{H}_j$ such that $A_j x = 0$. Also, there exists $y \in \mathcal{H}$ such that $y \in \ker(W^n)$, for some $n \geq 2$, but $y \notin \ker(W)$. We can write $y = (y_1, y_2, \ldots)$ where $y_k \in \mathcal{H}_k$, for all $k \in \mathbb{N}$, and we know that there exists $i \in \mathbb{N}$ such that $y_i \notin \ker(A_j)$ because $y \notin \ker(W)$. However, $W^n y = 0$ implies that $A_{i+n-1}A_{i+n-2}\cdots A_i y_i = 0$. By Lemma 5.1, $y_i \otimes x \in \mathcal{B}_W(i, j)$.

On the other hand, there does not exist $M > 0$ such that $||W^n (y_i \otimes x) z|| \leq M ||W^n z||$, for all $n \in \mathbb{N}$ and for all $z \in \mathcal{H}$. This is indeed the case because if $z = (z_1, z_2, \ldots)$ is the vector defined by $z_k = 0$, for $k \neq j$, and $z_j = x$, then $||W(y_i \otimes x) z|| = |\langle x, x \rangle| ||A_j y_i|| \neq 0$ but $||W z|| = ||A_j x|| = 0$. Thus, $y \otimes x \notin \mathcal{D}_W(i, j)$. Since $W$ has finite multiplicity, $\mathcal{D}_W(i, j)$ is weakly closed and it follows that $\overline{y \otimes x}$ does not belong to the weak closure of $\mathcal{D}_W$. Therefore, $\mathcal{B}_W$ properly contains the weak closure of $\mathcal{D}_W$. □

We now list another corollary to Lemma 5.1

Corollary 5.3 Let $W \sim (A_n)$ be a weighted shift of finite multiplicity with $r(W) > 0$ and let $i \in \mathbb{N}$. If there exists $n \in \mathbb{N}$ such that $A_{i+n-1}A_{i+n-2}\cdots A_i = 0$, then $\mathcal{B}(i, j) = \mathcal{L}^{\infty}(\mathcal{H}_j, \mathcal{H}_i)$, for all $j \in \mathbb{N}$.
Proof: Let $i, j \in \mathbb{N}$, let $y \in \mathcal{H}_i$, and let $x \in \mathcal{H}_j$. Assume that there exists $n \in \mathbb{N}$ such that $A_{i+n-1}A_{i+n-2} \cdots A_i = 0$. By Lemma 5.1, $y \otimes x \in B_W(i, j)$ and it follows that $B_W(i, j)$ contains all finite rank operators. Therefore, $B_W(i, j) = \mathcal{L}(\mathcal{H}_j, \mathcal{H}_i)$ because $W$ has finite multiplicity.

We now give an example of a weighted shift of multiplicity two such that there exists a n.i.s. for $B_W$ but $B_W(1, 1) = \mathcal{L}(\mathcal{H}_1)$ and $W$ is not quasinilpotent.

Example Define $W \sim (A_n)$ by
\[
A_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \quad \text{and} \quad A_n = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}, \quad \text{for} \quad n \geq 3.
\]
From this definition, one can quickly see that $A_2A_1 = 0$ and $\|W\|^n = 2^n$, for all $n \in \mathbb{N}$. Thus, $r(W) = 2$ and $B_W(1, 1) = \mathcal{L}(\mathcal{H}_1)$ by Corollary 5.3.

Before we construct a n.i.s. for $B_W$, we first prove that $B_W(3, 3) \subset \{ \begin{bmatrix} a \\ b \\ c \end{bmatrix} : a, b, c \in \mathbb{C} \}$.

Let $T = \begin{bmatrix} t_1 & t_2 \\ t_3 & t_4 \end{bmatrix} \in B_W(3, 3)$.

By Proposition 3.1, there exists $M > 0$ such that
\[
\sum_{n=0}^{\infty} d_m^{2n} \| A_{n+2}A_{n+1} \cdots A_3Tx \|^2 \leq M \sum_{n=0}^{\infty} d_m^{2n} \| A_{n+2}A_{n+1} \cdots A_3x \|^2,
\]
for all $x \in \mathcal{H}_3$ and for all $m \in \mathbb{N}$. Consider $x = (0, 1)^T$. The left hand side of (15) can now be written as
\[
\sum_{n=0}^{\infty} d_m^{2n} \| A_{n+2}A_{n+1} \cdots A_3x \|^2 = \sum_{n=0}^{\infty} \left( \frac{m}{1+2m} \right)^{2n} \left( |2^n t_2|^2 + |t_4|^2 \right)
\]
\[
= \frac{|t_2|^2}{1 - (\frac{2m}{1+2m})^2} + \frac{|t_4|^2}{1 - (\frac{m}{1+2m})^2}.
\]

Meanwhile, the right hand side of (15) simplifies to
\[
M \sum_{n=0}^{\infty} d_m^{2n} \| A_{n+2}A_{n+1} \cdots A_3x \|^2 = M \sum_{n=0}^{\infty} \left( \frac{m}{1+2m} \right)^{2n} = \frac{1}{1 - (\frac{m}{1+2m})^2}.
\]

Therefore, (15) becomes
\[
\frac{|t_2|^2}{1 - (\frac{2m}{1+2m})^2} + \frac{|t_4|^2}{1 - (\frac{m}{1+2m})^2} \leq \frac{M}{1 - (\frac{m}{1+2m})^2}.
\]

This implies that $|t_2| = 0$ because
\[
\lim_{m \to \infty} \frac{1}{1 - (\frac{2m}{1+2m})^2} = \infty, \quad \text{and} \quad \lim_{m \to \infty} \frac{1}{1 - (\frac{m}{1+2m})^2} = 2.
\]
Hence, operators in $B_W(3, 3)$ must have lower triangular matrices so there exists a n.i.s. $N_3$ for $B_W(3, 3)$. From here we proceed just as in the proof of Theorem 4.6. Namely, define $N_k$ to be the span of $\{Ax : A \in B_W(k, 3), x \in N_3\}$, for $k \neq 3$, and define $N$ to be the closure of $\oplus_{k \in \mathbb{N}} N_k$. Repeating the proof of Theorem 4.6, we obtain that $N$ is a n.i.s. for $B_W$.

Our next example features a shift that, at first glance, should not behave too differently from the previous one. Namely, they are both noninjective weighted shifts of multiplicity two with positive spectral radii such that $B_W(1, 1) = \mathcal{L}(H_1)$. However, there does not exist a n.i.s. for $B_W$ in the following example.

**Example** Let

$$X = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad Y = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.$$  

Define the weight sequence for $W \sim (A_n)$ by

$$A_n = \begin{cases} X & \text{for } n = 2^k, k \geq 1, \\ Y & \text{for } n = 2^k + 1, k \geq 1, \\ I & \text{otherwise}, \end{cases}$$

where $I$ is the $2 \times 2$ identity matrix. This sequence contains arbitrarily long finite sub-sequences of the form $(I, I, I, \ldots, I)$ which imply that $\|W^n\| \geq 1$, for all $n \in \mathbb{N}$. Since $\|W\| = 1$, we actually have that $\|W^n\| = 1$, for all $n \in \mathbb{N}$, and the spectral radius formula tells us that $r(W) = 1$. For every $k \in \mathbb{N}$, there exists $n_k$ such that $A_{k+n_k}A_{k+n_k-1} \cdots A_k \neq 0$ because $XY = YX = 0$. By Corollary 5.3, we can conclude that $B_W(k, j) = \mathcal{L}(H_j, H_k)$, for all $j, k \in \mathbb{N}$. Therefore, $B_W$ is weakly dense in $\mathcal{L}(H)$ and $B_W$ does not have a n.i.s.

The last two examples show that, when $W$ is a noninjective weighted shift and $r(W) > 0$, the equality $B_W(1, 1) = \mathcal{L}(H_1)$ is not the proper condition to guarantee that $B_W$ does not have a n.i.s. We will give a necessary and sufficient condition for $B_W$ to have a n.i.s. in Theorem 5.7 after we conduct a study of noninjective, quasinilpotent weighted shifts.

### 5.2 Noninjective quasinilpotent weighted shifts

In this section, we turn our attention to the study of weighted shifts which are noninjective and quasinilpotent. Let $x \in H_j$, for some $j \in \mathbb{N}$. We will use $n_x$ to denote the smallest positive integer such that $A_{j+n-1}A_{j+n-2} \cdots A_j x = 0$ and we will say that $n_x = \infty$ if $A_{j+n-1}A_{j+n-2} \cdots A_j x \neq 0$, for all $n \in \mathbb{N}$. We start off with a general result which applies to all noninjective quasinilpotent weighted shifts, even those of infinite multiplicity.

**Proposition 5.4** Let $W$ be a noninjective quasinilpotent weighted shift and let $T \in B_W(i, j)$. Then $n_T x \leq n_x$, for all $x \in H_j$. 

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Proof: If \( n_x = \infty \), then this inequality trivially holds. Let \( n_x \in \mathbb{N} \) and let \( T \in \mathcal{B}_W(i,j) \).

By Proposition 3.1 there exists \( M > 0 \) such that

\[
\sum_{n=0}^{\infty} d_m^{2n} \| A_{i+n-1}A_{i+n-2} \cdots A_1Tx \|^2 \leq M \sum_{n=0}^{\infty} d_m^{2n} \| A_{j+n-1}A_{j+n-2} \cdots A_jx \|^2, \tag{16}
\]

for all \( m \in \mathbb{N} \) and for all \( x \in \mathcal{H}_j \). Since \( W \) is quasinilpotent, \( d_m = m \) and (16) implies that

\[
\sum_{n=0}^{n_Tx} m^{2n} \| A_{i+n-1}A_{i+n-2} \cdots A_1Tx \|^2 \leq M \sum_{n=0}^{n_x} m^{2n} \| A_{j+n-1}A_{j+n-2} \cdots A_jx \|^2, \tag{17}
\]

for all \( m \in \mathbb{N} \). Both sides of (17) are polynomials in \( m \), so \( n_x \) cannot be strictly less than \( n_{T_x} \).

\[ \square \]

From here, we get an important result about the existence of a n.i.s. for \( \mathcal{B}_W \).

**Corollary 5.5** Let \( W \) be a noninjective quasinilpotent weighted shift. Then \( \ker(W) \) is a n.i.s. for \( \mathcal{B}_W \).

Proof: Let \( T \in \mathcal{B}_W \) and let \( \{ T_{ij} \}_{i,j \in \mathbb{N}} \) be its matrix where \( T_{ij} \in \mathcal{B}_W(i,j) \). It suffices to show that \( \ker(W) \) is an invariant subspace for \( T_{ij} \), for all \( i,j \in \mathbb{N} \). Let \( x = (x_1, x_2, \ldots) \in \ker(W) \), with \( x_n \in \mathcal{H}_n \), for all \( n \in \mathbb{N} \). Then, \( n_{x_k} \leq 1 \), for all \( k \in \mathbb{N} \), because \( A_k x_k \) must be zero. By Proposition 5.4, \( n_{T_{ij}x_j} \leq n_{x_j} \leq 1 \) and it follows that \( W T_{ij}x = A_i T_{ij}x = 0 \). Hence, \( \ker(W) \) is invariant for \( T_{ij} \), for all \( i,j \in \mathbb{N} \). Therefore, \( \ker(W) \) is an invariant subspace for \( \mathcal{B}_W \) and it is nontrivial because \( W \) is a nonzero, noninjective operator.

\[ \square \]

We now know that there exists a n.i.s. for \( \mathcal{B}_W \) if \( W \) is a quasinilpotent, noninjective weighted shift. Earlier, Theorem 4.6 offered an analogous result for quasinilpotent, injective weighted shifts of finite multiplicity. Combining these facts, we obtain the following theorem.

**Theorem 5.6** Let \( W \) be a quasinilpotent weighted shift of finite multiplicity. Then \( \mathcal{B}_W \) has a n.i.s.

Finally, we are in position to establish our main result, a characterization of weighted shifts of finite multiplicity such that the associated Spectral radius algebra possesses a n.i.s.

**Theorem 5.7** Let \( W \sim (A_n) \) be a weighted shift of finite multiplicity. Then there exists a n.i.s. for \( \mathcal{B}_W \) if and only if \( r(W) = 0 \) or \( \mathcal{B}_W(n,n) \neq \mathcal{L}(\mathcal{H}_n) \), for some \( n \in \mathbb{N} \).

Proof: We start off by showing that if \( r(W) > 0 \) and \( B_W(n,n) = \mathcal{L}(\mathcal{H}_n) \), for all \( n \in \mathbb{N} \), then \( \mathcal{B}_W \) does not have a n.i.s. More precisely, we will show that if \( \mathcal{M} \neq \{0\} \) is an invariant subspace for \( \mathcal{B}_W \), then \( \mathcal{M} = \mathcal{H} \). Since the projections on \( \mathcal{H}_k \) belong to \( \mathcal{B}_W \), for all \( k \in \mathbb{N} \), \( \mathcal{M} \) is necessarily of the form \( \mathcal{M} = \bigoplus_{k \in \mathbb{N}} \mathcal{M}_k \) where \( \mathcal{M}_k \subseteq \mathcal{H}_k \) is invariant for \( \mathcal{B}_W(k,k) = \mathcal{L}(\mathcal{H}_k) \). If we assume that \( \mathcal{M} \) is nonzero, then there exists \( n \in \mathbb{N} \) such that \( \mathcal{M}_n \neq 0 \), thus \( \mathcal{M}_n = \mathcal{H}_n \).
Since $A_n \in \mathcal{B}_W(n+1,n)$, $A_n x \in \mathcal{M}_{n+1}$, for all $x \in \mathcal{M}_n = \mathcal{H}_n$. Therefore, $\mathcal{M}_{n+1} \neq 0$ because $A_n \neq 0$ and $\mathcal{M}_{n+1} = \mathcal{H}_{n+1}$. This shows that $\mathcal{M}_k = \mathcal{H}_k$, for all $k \geq n$, and it remains to establish that $\mathcal{M}_k = \mathcal{H}_k$, for $k < n$.

If $A_{n-1}$ is injective, then $A_{n-1}^{-1}$ exists because $\dim(\mathcal{H}_n) = \dim(\mathcal{H}_{n-1}) < \infty$. Furthermore, we assumed that $r(W) > 0$, so Theorem 3.4 implies that $A_{n-1}^{-1} \in \mathcal{B}_W(n-1,n)$. Hence, $A_{n-1}^{-1} x \in \mathcal{M}_{n-1}$, for all $x \in \mathcal{M}_n = \mathcal{H}_n$, and $\mathcal{M}_{n-1} = \mathcal{H}_{n-1}$. On the other hand, if $A_{n-1}$ is not invertible, let $y$ be a nonzero vector in ker($A_{n-1}$). By Lemma 5.1, $y \otimes x \in \mathcal{B}_W(n-1,n)$, for all $x \in \mathcal{H}_n$, and it follows that $(y \otimes x)x = \|x\|^2 y \in \mathcal{M}_{n-1}$. Therefore, $\mathcal{M}_{n-1} \neq 0$ and $\mathcal{M}_{n-1} = \mathcal{H}_{n-1}$ because $\mathcal{M}_{n-1}$ is invariant for $\mathcal{B}_W(n-1,n-1) = L(\mathcal{H}_n)$. This shows that $\mathcal{M}_k = \mathcal{H}_k$, for all $k \in \mathbb{N}$, so $\mathcal{M} = \mathcal{H}$ and all invariant subspaces for $\mathcal{B}_W$ must be trivial.

In the other direction, we assume that $r(W) = 0$ or $\mathcal{B}_W(n,n) \neq L(\mathcal{H}_n)$ for some $n \in \mathbb{N}$ and we will prove that $\mathcal{B}_W$ has a n.i.s. If $r(W) = 0$, then this immediately follows from Theorem 5.6. If $\mathcal{B}_W(n,n) \neq L(\mathcal{H}_n)$, for some $n \in \mathbb{N}$, then there exists a n.i.s. $\mathcal{M}_n$ for $\mathcal{B}_W(n,n)$. We define $\mathcal{M}_k$ to be the span of $\{Ax : A \in \mathcal{B}_W(k,n), x \in \mathcal{M}_n\} \subset \mathcal{H}_k$, for all $k \in \mathbb{N}$, and define $\mathcal{M}$ to be the closure of $\bigoplus_{k \in \mathbb{N}} \mathcal{M}_k$. One can now quickly prove that $\mathcal{M}$ is a n.i.s. for $\mathcal{B}_W$ by repeating the proof of Theorem 4.6.

This answers the question about the existence of a n.i.s. for $\mathcal{B}_W$. One thing that is left unanswered is which algebras can actually be viewed as the Spectral radius algebra associated to some weighted shift. When investigating the Deddens algebra in [11], it was shown that, for weighted shifts of multiplicity two, one can construct a weighted shift $W_k$ such that $\dim(D_{W_k}(i,j)) = k$, for all $i, j \in \mathbb{N}$, where $1 \leq k \leq 4$. However, these examples had the property that $\dim(B_{W_k}(i,j))$ was either always three or always four. This leaves us with the following open problems.

**Problem 5.8** Does there exist a weighted shift of multiplicity two such that $\mathcal{B}_W(i,j)$ is one dimensional, for all $i, j \in \mathbb{N}$?

**Problem 5.9** Does there exist a weighted shift of multiplicity two such that $\mathcal{B}_W(i,j)$ is two dimensional, for all $i, j \in \mathbb{N}$?

Further in this direction, it was demonstrated in [11] that weighted shifts of multiplicity two had the property that changes in dimension of $D_W(i,j)$ can happen almost arbitrarily. However, it is not clear whether this carries over to Spectral radius algebras. In particular, every quasinilpotent weighted shift of multiplicity two that we considered had the property that if $\mathcal{B}_W(1,2)$ was one dimensional, then $\mathcal{B}_W(1,1)$ must be three dimensional. Taking into account Corollary 4.5, which states that $\dim(B_W(1,2)) < \dim(B_W(1,1))$, this leaves us with the following question.

**Problem 5.10** Does there exist an injective quasinilpotent weighted shift of multiplicity two such that $\dim(B_W(1,2)) = 1$ and $\dim(B_W(1,1)) = 2$?
If so, this would be a step in the right direction to showing that Spectral radius algebras associated to weighted shifts come in a wide variety. One last problem of interest is the subject of weighted shifts of infinite multiplicity.

**Problem 5.11** Can Theorem 4.6 be generalized so that it applies to injective weighted shifts of infinite multiplicity?

**References**


