Exam II Review

The scope of the exam will include:

1. **Applications of Integration** Chapter 6 including sections 6.1–6.5 and section 6.8
2. **Sequences** Chapter 8 section 8.1 only.

Chapter 6: Applications of Integration

Area Between Curves

Given two functions $f(x)$ and $g(x)$ such that $g(x) \leq f(x)$ on the interval $a \leq x \leq b$, the **area between** the graphs of $y = f(x)$ and $y = g(x)$ between $x = a$ and $x = b$ is

$$A = \int_a^b [f(x) - g(x)]\ dx.$$

Volumes

Always be sure to carefully consider the solid of interest to determine whether it will be easier to integrate with respect to x or y. In the descriptions below, it is assumed that the integration will be with respect to x. When integrating with respect to y, simply interchange the roles of x and y.

Disk method Given a function $y = f(x)$, if we spin it about the x-axis we obtain a solid of **volume**

$$V = \int_a^b \pi[f(x)]^2\ dx.$$

Washer method When there are "holes" in the solid of revolution, we use the washer method

$$V = \int_a^b \pi \left([f(x)]^2 - [g(x)]^2\right)\ dx$$

where $f(x)$ is function determining the outside radius and $g(x)$ the inside radius. When the revolution is carried out around a line other than the x- or y- axis, care must always be taken to determine the appropriate radius function(s).

Method of Cylindrical Shells For volume problems for which neither the method of disks or washers applies, consider the method of cylindrical shells. The volume of a solid obtained by rotating about the y-axis the region under the curve $y = f(x)$ from a to b is

$$V = \int_a^b 2\pi x f(x)\ dx$$

When the revolution is carried out around a line other than the x- or y- axis, care must always be taken to determine the appropriate radius function(s). In more complicated situations (e.g., torus), care must also be taken to ensure that the height function is correctly determined.

Arc Length

We have the formulas for **arc length** of a curve

$$L = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx,$$

$$L = \int_a^b \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy,$$

$$L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt,$$

where we denote the expression to the right of the integral sign by ds, the differential of arc length. Here are some guidelines for deciding which version of ds to use:

- If the problem is written in terms of parametric equations, then use $ds = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$.
- If the problem is given in terms of rectangular coordinates x and y, then:

  1. if $y = f(x)$ and you can't solve for x (e.g., $y = 2x - x^2$), then use $ds = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$
  2. if $x = g(y)$ and you can't solve for y (e.g., $x = \ln|\sec y|$), then use $ds = \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy$;
  3. if the equation may be expressed giving y as a function of x, $y = f(x)$, and x as a function of y, $x = g(y)$ (e.g., $2y = 3x^{2/3}$), then try both formulas and see which is simpler.

In all these problems, your first priority should be to simplify the expression under the square root, since this is where the most difficulties arise. You should expect to have to use techniques developed in chapter 5 to evaluate some of these integrals (e.g., trig substitution).

Average Value of a Function

The **average value** of a function $f(x)$ over the closed interval $[a, b]$ is

$$f\text{ave} = \frac{1}{b-a}\int_a^b f(x)\, dx.$$

Probability

The probability density function f models the probability that X lies between a and b by

$$P(a \le X \le b) = \int_a^b f(x)\, dx.$$

The probability density function of a random variable X must satify the conditions $f(x) \ge 0$ and

$$\int_a^b f(x)\, dx = 1,$$

since probabilities are measured on a scale from 0 to 1.

The **mean** of a random variable X with probability density function f is given by

$$\mu = \int_{-\infty}^{\infty} x f(x)\, dx,$$

while the **median** is the number m such that

$$\int_{-\infty}^{m} f(x)\, dx = \int_{m}^{\infty} f(x)\, dx = \frac{1}{2}.$$

The probability density function most commonly used to model a *waiting time* X is the **exponential density**

$$f(x) = \begin{cases} 0 & x < 0 \\ ce^{-cx} & x \geq 0. \end{cases}$$

The mean $\mu = 1/c$ of the exponential density is the reciprocal of the parameter c.

The most important distribution of all is the **normal distribution** with probability density function

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-(x-\mu)^2/(2\sigma^2)}$$

The density is defined for $-\infty < X < \infty$ and has parameters mean μ and standard deviation σ. Because the normal density does not have an elementary antiderivative, probabilities associated with the normal distribution are computed approximately via numerical integration on the calculator, an applet, or specialized tools for the normal.

Section 8.1: Sequences

A **sequence** is a function $f : \{1, 2, 3, \ldots\} \to \mathbb{R}$, which we often think of as an ordered list of real numbers $a_1, a_2, a_3, \ldots, a_n, \ldots$, where $a_n = f(n)$. We say that a sequence **converges to the limit** L if the terms a_n get arbitrarily close to L as $n \to \infty$. Sometimes the terms of a sequence are determined by an explicit formula, but sometimes they are defined *recursively* by specifying the value of a_1 and then indicating how to obtain a_{n+1} from a_n. In this case, if $\lim a_n = L$ and $a_{n+1} = f(a_n)$, then we must have that $L = f(L)$, which enables us to find the value of the limit if we first know that the limit exists. To answer this question, we often use the Monotonic Sequence Theorem, which states that a bounded, monotonic sequence converges.