Midterm I: Sample Solutions

1.

A. $\int_{-2}^{2} f(t)\,dt =$ area of semicircle from -2 to 0 and the half triangle from 0 to 2

$= \int_{-2}^{0} f(t)\,dt + \int_{0}^{2} f(t)\,dt$

$\int_{-2}^{0} f(t)\,dt = \frac{-\pi \cdot 1^2}{2} = \frac{-\pi}{2}$

$\int_{0}^{2} f(t)\,dt = \frac{1}{2} \cdot 2 \cdot 3 = 3$

$\int_{-2}^{0} f(t)\,dt + \int_{0}^{2} f(t)\,dt = \boxed{-\frac{\pi}{2} + 3}$

B. $\int_{-2}^{-4} f(t)\,dt = -\int_{-4}^{-2} f(t)\,dt$

$=$ Area of triangle from $t = [-4, -2]$

$= -\left(-\frac{1}{2} \cdot 2 \cdot 2\right) = 2$

$\boxed{\int_{-2}^{-4} f(t)\,dt = 2}$

C. $F(-3) = \int_{-2}^{-3} f(t)\,dt \rightarrow F'(-3) = f(-3)$

$f(-3) = -1 \rightarrow \boxed{F'(-3) = -1}$

1. D.

F is concave down whenever F' is decreasing.

$F' = f(x)$, therefore, whenever $f(x)$ is decreasing, F is concave down.

The intervals where F is concave down are:

$\boxed{x = [-2, -1], [2, 4]}$

2. a.) $\int_0^2 3x^2\,dx$

$\Delta x = \frac{b-a}{n} = \frac{2-0}{n} = \frac{2}{n}$

$x_i = a + i\Delta x = 0 + \frac{2i}{n} = \frac{2i}{n}$

$\lim_{n\to\infty} \sum_{i=1}^{n} 3(x_i)^2 \Delta x$

$\lim_{n\to\infty} \sum_{i=1}^{n} \left[3\left(\frac{2i}{n}\right)^2\right]\frac{2}{n}$

$\lim_{n\to\infty} 3 \sum_{i=1}^{n} \left(\frac{4}{n^2} i^2\right)\frac{2}{n}$

$$\lim_{n\to\infty} 3 \cdot \left[\frac{8}{n^3} \cdot \frac{n(n+1)(2n+1)}{6}\right]$$

$$\lim_{n\to\infty} 3\left[\frac{8}{6} \cdot \frac{n(n+1)(2n+1)}{n^3}\right]$$

$$\lim_{n\to\infty} 3\left[\frac{8}{6}\cdot\left(\frac{n}{n}\right)\left(\frac{n+1}{n}\right)\left(\frac{2n+1}{n}\right)\right]$$

$$\lim_{n\to\infty} 3\left[\frac{8}{6}(1)\left(\frac{n}{n}+\frac{1}{n}\right)\left(\frac{2n}{n}+\frac{1}{n}\right)\right]$$

$$\lim_{n\to\infty} 3\left(\frac{8}{6}\right)(1)\left(1+\frac{1}{n}\right)\left(2+\frac{1}{n}\right)$$

$$3\left(\frac{8}{6}\right)(1)(1)(2) = \boxed{8}$$

b.) $\int_0^2 3x^2\,dx \quad ; \quad F'(x) = f(x)$

$$3\int_0^2 x^2\,dx$$

$$3\left(\frac{x^3}{3}\right)\Big]_0^2$$

$$x^3\Big]_0^2$$

$$2^3 - 0^3 = \boxed{8}$$

3

a) $\frac{1}{4-u^2}\,du$

b) $\int \frac{1}{(2+u)(2-u)}\,du$

$\int \frac{A}{2+u} + \frac{B}{2-u}$

$2B + 2A = 1$

$uB - uA = 0$

$uB = uA$

$B = A$

$2B + 2B = 1$

$B = 1/4$

$A = 1/4$

$\int \frac{1}{4(2+u)}\,du + \int \frac{1}{4(2-u)}\,du$

$\frac{1}{4}\int \frac{1}{2+u}\,du + \frac{1}{4}\int \frac{1}{2-u}\,du$

$v = 2+u$ $\quad$ $v = 2-u$

$dv = 1 \cdot du$ $\quad$ $dv = -du$

$\frac{1}{4}\ln|2+u| - \frac{1}{4}\ln|2-u|$

$$\boxed{\frac{1}{4}\ln\left|\frac{2+\sin(x)}{2-\sin(x)}\right| + C}$$

$$\frac{1}{4}\ln|2+u| - \frac{1}{4}\ln|2-u|$$

$$\boxed{\frac{1}{4}\ln\left|\frac{2+\sin(x)}{2-\sin(x)}\right| + C}$$

4a) let f be any function that is differentiable with derivative f'

using integration by parts

$$\int x f'(x)dx = xf(x) - \int f(x)\,dx \qquad u = x \quad v = f(x)$$
$$u' = 1 \quad v' = f'(x)$$

$\therefore$ The formula is true for any differentiable function f.

b) let $f'(x) = \dfrac{1}{\sqrt[3]{x-1}} = (x-1)^{-\frac{1}{3}}$

then $f(x) = \dfrac{3(x-1)^{\frac{2}{3}}}{2}$

$$\int \frac{x}{\sqrt[3]{x-1}}\,dx = \frac{3x(x-1)^{\frac{2}{3}}}{2} - \frac{3}{2}\int (x-1)^{\frac{2}{3}}\,dx$$

$$= \frac{3x(x-1)^{\frac{2}{3}}}{2} - \frac{3}{2}\cdot\frac{3}{5}\cdot(x-1)^{\frac{5}{3}} + C$$

$$= \frac{3x(x-1)^{\frac{2}{3}}}{2} - \frac{9}{10}(x-1)^{\frac{5}{3}} + C$$

5 a) $\Gamma(x) = \int_0^\infty t^{x-1} e^{-t}\,dt$

$\Gamma(1) = \int_0^\infty t^{(1-1)} e^{-t}\,dt = \int_0^\infty t^0 e^{-t}\,dt = \lim_{w\to\infty} \int_0^w e^{-t}\,dt =$

$\lim_{w\to\infty} -\left[e^{-t}\right]_0^w = \lim_{w\to\infty} -\left(\cancel{e^{-w}}^{0} - e^0\right) = 1$

b) $\Gamma(2) = \int_0^\infty t^{(2-1)} e^{-t}\,dt = \int_0^\infty t e^{-t}\,dt = \lim_{w\to\infty} \int_0^w t e^{-t}\,dt$ | $u = t \quad dv = e^{-t}dt$

$= \lim_{w\to\infty} -\left(\left[t e^{-t}\right]_0^w + \int_0^w e^{-t}\,dt\right) = \lim_{w\to\infty} -\left(\frac{w}{e^w} + \left[e^{-t}\right]_0^w\right)$ | $du = dt \quad v = e^{-t}$

$= \lim_{w\to\infty} -\left(\frac{w}{e^w} + \left(\frac{1}{e^w} - e^0\right)\right) = \lim_{w\to\infty} -\left(\cancel{\frac{w}{e^w}}^{0} + \cancel{\frac{1}{e^w}}^{0} - 1\right) = 1$

6. a) $\int \frac{4x}{\sqrt{25-x^2}}\,dx \quad u = 25 - x^2 \quad du = -2x\,dx$

$-2\int \frac{1}{\sqrt{u}}\,du = -2\cdot 2\sqrt{u}$

$\boxed{-4\sqrt{25-x^2} + C}$

b) $\int \frac{4}{\sqrt{25-x^2}}\,dx = 4\int \frac{1}{\sqrt{25-x^2}}\,dx$

$\boxed{4\cdot \arcsin\left(\frac{x}{5}\right) + C}$

c) $\int \frac{x\cos(x^2)}{\sqrt{\sin(x^2)}}\,dx \quad u = x^2 \quad du = 2x\,dx$

$$\frac{1}{2}\int \frac{\cos(u)}{\sqrt{\sin(u)}}\,du = \frac{1}{2}\cdot 2\cdot\sqrt{\sin(u)}$$

$$\boxed{\sqrt{\sin(x^3)} + C}$$

d) $\int \frac{2\ln x}{x^3}dx = 2\int \frac{\ln x}{x^3}\,dx$

$dv = x^{-3}dx \quad v = \frac{1}{-2x^2}$

$u = \ln x \quad du = \frac{1}{x}dx$

$$\frac{2\ln x}{-2x^2} - 2\int \frac{1}{-2x^2}\cdot\frac{1}{x}\,dx =$$

$$\frac{-\ln x}{x^2} - \int \frac{-1}{x^3} = \frac{-\ln x}{x^2} - \left(\frac{1}{2x^2}\right) + C$$

$$\boxed{\frac{-\ln x}{x^2} - \frac{1}{2x^2} + C}$$

e) $\frac{1}{\pi}\int \ln x\,dx \quad du = dx \quad v = x$

$u = \ln x \quad du = \frac{1}{x}dx$

$$\frac{x\ln x}{\pi} - \frac{1}{\pi}\int \frac{\cancel{x}}{\cancel{x}}\,dx = \frac{x\ln x - x}{\pi} = \boxed{\frac{x(\ln x - 1)}{\pi} + C}$$

F) $\int \frac{\sin\theta}{\cos\theta}\, d\theta$ $\quad u = \cos\theta$

$du = -\sin\theta$

$-\int \frac{1}{u}\, du = \boxed{-\ln|\cos\theta| + C}$

Bonus Question

$$\int \frac{4x^2}{\sqrt{25-x^2}}\,dx \qquad \text{let } x = 5\sin\theta$$

$$\frac{dx}{d\theta} = 5\cos\theta$$

$$= \int \frac{100\sin^2\theta}{\sqrt{25-25\sin^2\theta}} \cdot 5\cos\theta \, d\theta$$

$$= \int \frac{100\sin^2\theta}{5\cos\theta} \cdot 5\cos\theta \, d\theta$$

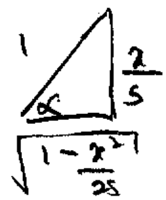
$$= 100\int \sin^2\theta \, d\theta \qquad \sin^2\theta = \frac{1}{2}(1-\cos 2\theta)$$

$$= 50\int 1-\cos 2\theta \, d\theta$$

$$= 50\left[\theta - \frac{\sin 2\theta}{2}\right] + C \qquad \theta = \arcsin\left(\frac{x}{5}\right)$$

$$= 50\theta - 25\sin 2\theta + C$$

$$= 50\arcsin\left(\frac{x}{5}\right) - 25\sin\left(2\arcsin\left(\frac{x}{5}\right)\right) + C$$



$= 50\arcsin\left(\frac{x}{5}\right) - 50\sin\left(\arcsin\left(\frac{x}{5}\right)\right)\cos\left(\arcsin\left(\frac{x}{5}\right)\right) + c$

$= 50\arcsin\left(\frac{x}{5}\right) - 50 \cdot \frac{x}{5} \cdot \sqrt{1-\frac{x^2}{25}} + C$

$= 50\arcsin\left(\frac{x}{5}\right) - 10x\sqrt{1-\frac{x^2}{25}} + C$