Final Exam Review

The scope of the final exam will include:

1. **Integrals** Chapter 5 including sections 5.1–5.7, 5.10
2. **Applications of Integration** Chapter 6 including sections 6.1–6.5 and section 6.8
3. **Infinite Sequences and Series** Chapter 8 including sections 8.1–8.4; sections 8.5–8.7 limited to lecture material.

Chapter 5: Integrals

Chapter 5 is an introduction to integral calculus. In this chapter, we studied the definition of the integral in terms of Riemann sums and we developed the notions of definite and indefinite integrals. Later, we studied techniques for evaluating both of these kinds integrals.

Sigma Notation

A convenient shorthand for writing long (and even infinite) sums is the so-called "sigma notation." The expression $a_1 + a_2 + \cdots + a_n$ is abbreviated $\sum_{i=1}^{n} a_i$ *i.e.*,

$$\sum_{i=1}^{n} a_i = a_1 + a_2 + \cdots + a_n,$$

but this representation of the sum is not unique. There are important **formulas** to know:

- $\sum_{i=1}^{n} 1 = n$,
- $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$,
- $\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$,
- $\sum_{i=1}^{n} i^3 = \left(\frac{n(n+1)}{2}\right)^2$,

and **rules**

- $\sum_{i=1}^{n} kf(i) = k\sum_{i=1}^{n} f(i)$,
- $\sum_{i=1}^{n}[f(i) \pm g(i)] = \sum_{i=1}^{n} f(i) \pm \sum_{i=1}^{n} g(i)$

Riemann Sums and Definite Integrals

Our introduction to definite integrals begins with Riemann sums. First, we divide the interval into $[a, b]$ into n subintervals, each of width

$$\Delta x = \frac{b-a}{n}.$$

Next, we determined the endpoints of these subintervals to be

$$x_i = a + i\Delta x.$$

for $j = 0, 1, 2, \ldots, n$. In particular, $x_0 = a$ and $x_n = b$. Then a **Riemann sum** for a function f on the interval $[a, b]$ is defined to be

$$\sum_{i=1}^{n} f(x_i^*)\Delta x = f(x_1^*)\Delta x + f(x_2^*)\Delta x + \ldots + f(x_n^*)\Delta x$$

where x_i is an arbitrary sample point in the i-th subinterval $[x_{i-1}, x_i]$ for $i = 1, 2, \ldots, n$. We defined the **definite integral on** $[a, b]$ to be

$$\int_a^b f(x)\, dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x$$

We we compute definite integrals using the limit definition above, we often take x_i^* to be the right-hand endpoint, x_i, of the ith subinterval.

(Signed) area: The definite integral $\int_a^b f(t)\, dt$ is the *signed* area of the region in the plane bounded by the graph of $y = f(x)$ and the lines $y = 0$, $x = a$, and $x = b$. Remember that area *above* the x-axis is positive, but area *below* the x-axis is negative.

Properties of Definite Integrals:

- $\int_a^b [f(t) + g(t)]\, dt = \int_a^b f(t)\, dt + \int_a^b g(t)\, dt$ and $\int_a^b kf(t)\, dt = k\int_a^b f(t)\, dt$
- $\int_a^a f(t)\, dt = 0$; $\int_a^b f(t)\, dt = -\int_b^a f(t)\, dt$; and $\int_a^b f(t)\, dt = \int_a^c f(t)\, dt + \int_c^b f(t)\, dt$
- If $m \le f(x) \le M$ on $[a, b]$, then $m(b-a) \le \int_a^b f(x)\, dx \le M(b-a)$

Evaluation Theorem: Suppose f is continuous on $[a, b]$. Then

$$\int_a^b f(t)\, dt = F(b) - F(a),$$

where F is *any* antiderivative of f, (*i.e.*, $F'(x) = f(x)$).

The Fundamental Theorem of Calculus: If f is continuous on the interval $[a, b]$, then the function F defined by $F(x) = \int_a^x f(t)\, dt$ is differentiable on $[a, b]$ and

$$F'(x) = \frac{d}{dx}\left[\int_a^x f(t)\, dt\right] = f(x).$$

Indefinite Integrals

Integral Formulas: Know the integrals:

1. $\int x^n\, dx = \dfrac{u^{n+1}}{n+1} + C$, if $n \neq -1$
2. $\int \frac{1}{x}\, dx = \ln|x| + C$,
3. $\int e^x\, dx = e^x + C$,
4. $\int \sin x\, dx = -\cos x + C$,
5. $\int \cos x\, dx = \sin x + C$,
6. $\int \sec^2 x\, dx = \tan x + C$,

7. $\int \sec x \tan x \, dx = \sec x + C$,

8. $\int \dfrac{1}{1+x^2} \, dx = \tan^{-1} x + C$,

9. $\int \dfrac{1}{\sqrt{1-x^2}} \, dx = \sin^{-1} x + C$.

Techniques of Integration: This is what sections 5.5, 5.6 and 5.7 were all about. There are several techniques, so we need a way to decide which to use. Here is a strategy to use when the integral isn't in one of the basic forms above:

1. *Simplify:* multiply, cancel, use trigonometric identities, etc.
2. *Substitution:* is there an obvious substitution? If the integral has the form $\int f(g(x))g'(x) \, dx$, then use
$$\int f(g(x))g'(x) \, dx = \int f(u) \, du, \text{ where } u = g(x).$$
3. *Integration by Parts:* use $\int u \, dv = uv - \int v \, du$ to replace a difficult integral with something easier.
4. *Classify:* is the integral of one of the types we saw in Section 5.7?
    - trig functions only? try using trig identities
    - sum/difference of squares? radicals? consider a trig substitution
    - rational function? try separating the fraction into the sum of rational functions whose denominators are the factors of the original
5. *Be creative:* another substitution, parts more than once, manipulate the integrand (rationalize the denominator, identities, multiply by just the right form of "1"), or combine several methods.

Additional Techniques of Integration

Trigonometric Integrals

For integrals of the form $\int \sin^m(x) \cos^n(x) \, dx$, consider the following strategies:

- If m is odd, save one sine factor and use $sin^2 x = 1 - cos^2 x$ to express the remaining factors in terms of cosine.
- If n is odd, save one cosine factor and use $cos^2 x = 1 - sin^2 x$ to express the remaining factors in terms of sine.
- If both m and n are even, use the half-angle identities
$$\sin^2 x = \frac{1-\cos 2x}{2} \qquad \cos^2 x = \frac{1+\cos 2x}{2}$$
to try to simplify. It is often helpful to recall $\sin 2x = 2 \sin x \cos x$.

Trigonometric Substitutions

Changing the variable of an integrand from x to θ using a trigonometric substitution allows us to simplify the integration of forms with radicals that would otherwise not be amenable to ordinary substitutions. The trig substitutions with which you should be familiar are:

- For $\sqrt{a^2 - x^2}$ use $x = a \sin\theta$
- For $\sqrt{a^2 + x^2}$ use $x = a \tan\theta$
- For $\sqrt{x^2 - a^2}$ use $x = a \sec\theta$.

Partial Fractions

We integrate rational functions (ratios of polynomials) by expressing them as sums of simpler fractions, called *partial fractions* that we already know how to integrate. We have limited our consideration to partial fractions in which the denominator is a quadratic and the numerator is a linear form such as the following (from Example 4 of Section 5.7):

$$\int \frac{5x - 4}{2x^2 + x - 1}\, dx.$$

Hence, you should be able to re-express this problem in the form

$$\int \frac{A}{2x - 1} + \frac{B}{x + 1}\, dx$$

identifying A and B and completing the integration.

Improper Integrals

We have considered two types of improper integrals: integrals on infinite intervals and integrals with discontinuous integrands.

Type 1 Improper Integrals

Integrals on infinite intervals have the forms

$$\int_a^\infty f(x)\, dx = \lim_{t\to\infty} \int_a^t f(x)\, dx \text{ and } \int_{-\infty}^a f(x)\, dx = \lim_{t\to-\infty} \int_a^t f(x)\, dx$$

An improper integral is *convergent* if the limit exists and divergent otherwise.
If both $\int_a^\infty f(x)\, dx$ and $\int_{-\infty}^a f(x)\, dx$ are convergent, then we define

$$\int_{-\infty}^\infty f(x)\, dx = \int_{-\infty}^a f(x)\, dx + \int_a^\infty f(x)\, dx$$

Of particular interest is the integral $\int_1^\infty \frac{1}{x^p}\, dx$ which is convergent if $p > 1$ and divergent if $p \leq 1$.

Type 2 Improper Integrals

We considered three cases involving discontinuities. If f is continuous on $[a, b)$ and is discontinuous at b, then

$$\int_a^b f(x)\, dx = \lim_{t \to b^-} f(x)\, dx,$$

if this limit exists (as a finite number). Similarly,

$$\int_a^b f(x)\, dx = \lim_{t \to a^+} f(x)\, dx.$$

If f has a discontinuity at c, where $a < c < b$, and both $\int_a^c f(x)\, dx$ and $\int_c^b f(x)\, dx$ are convergent, then we define

$$\int_a^b f(x)\, dx = \int_a^c f(x)\, dx + \int_c^b f(x)\, dx.$$

Hence, an integral over an interval where f has a discontinuity will diverge if the integral over either half diverges.

Chapter 6: Applications of Integration

Area Between Curves

Given two functions $f(x)$ and $g(x)$ such that $g(x) \leq f(x)$ on the interval $a \leq x \leq b$, the **area between** the graphs of $y = f(x)$ and $y = g(x)$ between $x = a$ and $x = b$ is

$$A = \int_a^b [f(x) - g(x)]\; dx.$$

Volumes

Always be sure to carefully consider the solid of interest to determine whether it will be easier to integrate with respect to x or y. In the descriptions below, it is assumed that the integration will be with respect to x. When integrating with respect to y, simply interchange the roles of x and y.

Disk method Given a function $y = f(x)$, if we spin it about the x-axis we obtain a solid of **volume**

$$V = \int_a^b \pi [f(x)]^2\, dx.$$

Washer method When there are "holes" in the solid of revolution, we use the washer method

$$V = \int_a^b \pi \left([f(x)]^2 - [g(x)]^2 \right)\; dx$$

where $f(x)$ is function determining the outside radius and $g(x)$ the inside radius. When the revolution is carried out around a line other than the x- or y- axis, care must always be taken to determine the appropriate radius function(s).

Method of Cylindrical Shells For volume problems for which neither the method of disks or washers applies, consider the method of cylindrical shells. The volume of a solid obtained by rotating about the y-axis the region under the curve $y = f(x)$ from a to b is

$$V = \int_a^b 2\pi x f(x)\, dx$$

When the revolution is carried out around a line other than the x- or y- axis, care must always be taken to determine the appropriate radius function(s). In more complicated situations (e.g., torus), care must also be taken to ensure that the height function is correctly determined.

Arc Length

We have the formulas for **arc length** of a curve

$$L = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx,$$
$$L = \int_a^b \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy,$$
$$L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt,$$

where we denote the expression to the right of the integral sign by ds, the differential of arc length. Here are some guidelines for deciding which version of ds to use:

- If the problem is written in terms of parametric equations, then use $ds = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$.
- If the problem is given in terms of rectangular coordinates x and y, then:
  1. if $y = f(x)$ and you can't solve for x (e.g., $y = 2x - x^2$), then use $ds = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$
  2. if $x = g(y)$ and you can't solve for y (e.g., $x = \ln|\sec y|$), then use $ds = \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy$;
  3. if the equation may be expressed giving y as a function of x, $y = f(x)$, and x as a function of y, $x = g(y)$ (e.g., $2y = 3x^{2/3}$), then try both formulas and see which is simpler.

In all these problems, your first priority should be to simplify the expression under the square root, since this is where the most difficulties arise. You should expect to have to use techniques developed in chapter 5 to evaluate some of these integrals (e.g., trig substitution).

Average Value of a Function

The **average value** of a function $f(x)$ over the closed interval $[a, b]$ is

$$f\text{ave} = \frac{1}{b-a} \int_a^b f(x)\, dx.$$

Probability

The probability density function f models the probability that X lies between a and b by

$$P(a \leq X \leq b) = \int_a^b f(x)\,dx.$$

The probability density function of a random variable X must satify the conditions $f(x) \geq 0$ and

$$\int_a^b f(x)\,dx = 1,$$

since probabilities are measured on a scale from 0 to 1.

The **mean** of a random variable X with probability density function f is given by

$$\mu = \int_{-\infty}^{\infty} x f(x)\,dx,$$

while the **median** is the number m such that

$$\int_{-\infty}^{m} f(x)\,dx = \int_{m}^{\infty} f(x)\,dx = \frac{1}{2}.$$

In more generality, we defined the pth quantile for a random variable as

$$\int_{-\infty}^{q_p} f(x)\,dx = p.$$

The probability density function most commonly used to model a *waiting time* X is the **exponential density**

$$f(x) = \begin{cases} 0 & x < 0 \\ \frac{1}{\beta} e^{-x/\beta} & x \geq 0. \end{cases}$$

The parameter β of the exponential density is also its mean.

The most important distribution of all is the **normal distribution** with probability density function

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-(x-\mu)^2/(2\sigma^2)}$$

The density is defined for $-\infty < X < \infty$ and has parameters mean μ and standard deviation σ.

Chapter 8: Infinite Sequences and Series

Sequences

A **sequence** is a function $f : \{1, 2, 3, \ldots\} \to \mathbb{R}$, which we often think of as an ordered list of real numbers $a_1, a_2, a_3, \ldots, a_n, \ldots$, where $a_n = f(n)$. We say that a sequence **converges to the limit** L if the terms a_n get arbitrarily close to L as $n \to \infty$. Sometimes the terms of a sequence are determined by an explicit formula, but sometimes they are defined *recursively* by specifying the value of a_1 and then indicating how to obtain a_{n+1} from a_n. In this case, if $\lim a_n = L$ and $a_{n+1} = f(a_n)$, then we must have that $L = f(L)$, which enables us to find the value of the limit if we first know that the limit exists. To answer this question, we often use the Monotonic Sequence Theorem, which states that a bounded, monotonic sequence converges.

Numerical Series

A **series** is obtained from a sequence $\{a_n\}$ by *adding* the terms: $\sum a_n = a_1 + a_2 + \cdots + a_n + \cdots$ is the series associated with the sequence $\{a_n\}$. While we cannot compute the infinite sum directly, we can compute the **partial sums** $s_n = a_1 + a_2 + \cdots + a_n$ and check for convergence of this new sequence $\{s_n\}$ as $\to \infty$. If it converges to the limit s, we say that the **series converges with sum** s, and write $\sum a_n = s$. If the sequence of partial sums fails to converge, we say that the series $\sum a_n$ diverges.

Below is a strategy for determining if a series converges:

1. Is the series one of the basic types?
   - **Geometric series:** $\sum_{n=1}^{\infty} ar^{n-1}$ or, equivalently, $\sum_{n=0}^{\infty} ar^n$, converges to $\frac{a}{1-r}$ when $|r| < 1$ and diverges when $|r| \geq 1$.
   - p**-Series**: $\sum_{n=1}^{\infty} \frac{1}{n_p}$ converges if $p > 1$ and diverges when $p \leq 1$.
2. **Test for Divergence:** If $\lim_{n\to\infty} a_n \neq 0$, then $\sum a_n$ diverges.
3. Classify:
   - does a_n involve factorials, products, and/or quotients? try the **Ratio Test** (RT)
   - is it alternating? use the **Alternating Series Test** (AST)
   - does it look like a geometric or p-series? try the **Comparison Test** (CT) or **Limit Comparison Test** (LCT);
   - $\sum f(n)$ converges if and only if $\int_1^{\infty} f(x)\, dx$ converges (**Integral Test**).

Power Series

Next we introduced power series, *i.e.*, series of the form $\sum_{n=0}^{\infty} = c_n x^n$. One important example of a power series that we studied is the **geometric power series**:

$$\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \cdots + x^n + \cdots$$

For x within the radius of convergence R, rules of polynomials also apply to power series, e.g., addition, multiplication, and term-by-term differentiation and integration.

The most important class of examples of power series are the Taylor series for functions. The **Taylor series of the function** f **about** a is

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n.$$

For the special case $a = 0$, we obtain the **Maclaurin series for** f **at** a:

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n.$$

You are responsible for knowing the following Maclaurin series, both of which are convergent for all real numbers x:

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

and

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$