Math 113: Review for Exam 1

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Chapter 5 is an introduction to integral calculus. In this chapter, we studied the definition of the integral in terms of Riemann sums and we developed the notions of definite and indefinite integrals. Later, we studied techniques for evaluating both of these kinds integrals.

1 Sigma Notation

A convenient shorthand for writing long (and even infinite) sums is the so-called "sigma notation." The expression $a_1 + a_2 + \cdots + a_n$ is abbreviated $\sum_{i=1}^{n} a_i$ *i.e.*,

$$\sum_{i=1}^{n} a_i = a_1 + a_2 + \cdots + a_n,$$

but this representation of the sum is not unique. There are important **formulas** to know:

- $\sum_{i=1}^{n} 1 = n,$
- $\sum_{i=1}^{n} i = \frac{n(n+1)}{2},$
- $\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6},$
- $\sum_{i=1}^{n} i^3 = \left(\frac{n(n+1)}{2}\right),$

and **rules**

- $\sum_{i=1}^{n} kf(i) = k\sum_{i=1}^{n} f(i),$
- $\sum_{i=1}^{n} [f(i) \pm g(i)] = \sum_{i=1}^{n} f(i) \pm \sum_{i=1}^{n} g(i)$

2 Riemann Sums and Definite Integrals

Our introduction to definite integrals begins with Riemann sums. First, we divide the interval into $[a, b]$ into n subintervals, each of width

$$\Delta x = \frac{b-a}{n}.$$

Next, we determined the endpoints of these subintervals to be

$$x_i = a + i\Delta x.$$

for $j = 0, 1, 2, \ldots, n$. In particular, $x_0 = a$ and $x_n = b$. Then a **Riemann sum** for a function f on the interval $[a, b]$ is defined to be

$$\sum_{i=1}^{n} f(x_i^*)\Delta x = f(x_1^*)\Delta x + f(x_2^*)\Delta x + \ldots + f(x_n^*)\Delta x$$

where x_i is an arbitrary sample point in the i-th subinterval $[x_{i-1}, x_i]$ for $i = 1, 2, \ldots, n$. We defined the **definite integral on** $[a, b]$ to be

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x$$

We we compute definite integrals using the limit definition above, we often take x_i^* to be the right-hand endpoint, x_i, of the ith subinterval.

(Signed) area: The definite integral $\int_a^b f(t)\,dt$ is the *signed* area of the region in the plane bounded by the graph of $y = f(x)$ and the lines $y = 0$, $x = a$, and $x = b$. Remember that area *above* the x-axis is positive, but area *below* the x-axis is negative.

Properties of Definite Integrals:

- $\int_a^b [f(t) + g(t)]\,dt = \int_a^b f(t)\,dt + \int_a^b g(t)\,dt$ and $\int_a^b kf(t)\,dt = k\int_a^b f(t)\,dt$
- $\int_a^a f(t)\,dt = 0$; $\int_a^b f(t)\,dt = -\int_b^a f(t)\,dt$; and $\int_a^b f(t)\,dt = \int_a^c f(t)\,dt + \int_c^b f(t)\,dt$
- If $m \le f(x) \le M$ on $[a, b]$, then $m(b-a) \le \int_a^b f(x)\,dx \le M(b-a)$

Evaluation Theorem: Suppose f is continuous on $[a, b]$. Then

$$\int_a^b f(t)\,dt = F(b) - F(a),$$

where F is *any* antiderivative of f, (*i.e.*, $F'(x) = f(x)$).

The Fundamental Theorem of Calculus: If f is continuous on the interval $[a, b]$, then the function F defined by $F(x) = \int_a^x f(t)\,dt$ is differentiable on $[a, b]$ and

$$F'(x) = \frac{d}{dx}\left[\int_a^x f(t)\,dt\right] = f(x).$$

3 Indefinite Integrals

Integral Formulas: Know the integrals:

1. $\int x^n\,dx = \dfrac{u^{n+1}}{n+1} + C$, if $n \neq -1$
2. $\int \frac{1}{x}\,dx = \ln|x| + C$,
3. $\int e^x\,dx = e^x + C$,
4. $\int \sin x\,dx = -\cos x + C$,
5. $\int \cos x\,dx = \sin x + C$,
6. $\int \sec^2 x\,dx = \tan x + C$,
7. $\int \sec x \tan x\,dx = \sec x + C$,
8. $\int \dfrac{1}{1+x^2}\,dx = \tan^{-1} x + C$,
9. $\int \dfrac{1}{\sqrt{1-x^2}}\,dx = \sin^{-1} x + C$.

Techniques of Integration: This is what sections 5.5, 5.6 and 5.7 were all about. There are several techniques, so we need a way to decide which to use. Here is a strategy to use when the integral isn't in one of the basic forms above:

1. *Simplify:* multiply, cancel, use trigonometric identities, etc.
2. *Substitution:* is there an obvious substitution? If the integral has the form $\int f(g(x))g'(x)\,dx$, then use
$$\int f(g(x))g'(x)\,dx = \int f(u)\,du, \text{ where } u = g(x).$$
3. *Integration by Parts:* use $\int u\,dv = uv - \int v\,du$ to replace a difficult integral with something easier.
4. *Classify:* is the integral of one of the types we saw in Section 5.7?
   - trig functions only? try using trig identities
   - sum/difference of squares? radicals? consider a trig substitution
   - rational function? try separating the fraction into the sum of rational functions whose denominators are the factors of the original
5. *Be creative:* another substitution, parts more than once, manipulate the integrand (rationalize the denominator, identities, multiply by just the right form of "1"), or combine several methods.

4 Additional Techniques of Integration

4.1 Trigonometric Integrals

For integrals of the form $\int \sin^m(x)\cos^n(x)\,dx$, consider the following strategies:

- If m is odd, save one sine factor and use $sin^2x = 1 - cos^2x$ to express the remaining factors in terms of cosine.
- If n is odd, save one cosine factor and use $cos^2x = 1 - sin^2x$ to express the remaining factors in terms of sine.
- If both m and n are even, use the half-angle identities
$$\sin^2 x = \frac{1-\cos 2x}{2} \qquad \cos^2 x = \frac{1+\cos 2x}{2}$$
to try to simplify. It is often helpful to recall $\sin 2x = 2\sin x\cos x$.

4.2 Trigonometric Substitutions

Changing the variable of an integrand from x to θ using a trigonometric substitution allows us to simplify the integration of forms with radicals that would otherwise not be amenable to ordinary substitutions. The trig substitutions with which you should be familiar are:

- For $\sqrt{a^2-x^2}$ use $x = a\sin\theta$
- For $\sqrt{a^2+x^2}$ use $x = a\tan\theta$
- For $\sqrt{x^2-a^2}$ use $x = a\sec\theta$.

4.3 Partial Fractions

We integrate rational functions (ratios of polynomials) by expressing them as sums of simpler fractions, called *partial fractions* that we already know how to integrate. We have limited our consideration to partial fractions in which the denominator is a quadratic and the numerator is a linear form such as the following (from Example 4 of Section 5.7):

$$\int \frac{5x-4}{2x^2+x-1}\,dx.$$

Hence, you should be able to re-express this problem in the form

$$\int \frac{A}{2x-1} + \frac{B}{x+1}\,dx$$

identifying A and B and completing the integration.

5 Improper Integrals

We have considered two types of improper integrals: integrals on infinite intervals and integrals with discontinuous integrands.

5.1 Type 1 Improper Integrals

Integrals on infinite intervals have the forms

$$\int_a^\infty f(x)\,dx = \lim_{t\to\infty}\int_a^t f(x)\,dx \text{ and } \int_{-\infty}^a f(x)\,dx = \lim_{t\to-\infty}\int_a^t f(x)\,dx$$

An improper integral is *convergent* if the limit exists and divergent otherwise.
If both $\int_a^\infty f(x)\,dx$ and $\int_{-\infty}^a f(x)\,dx$ are convergent, then we define

$$\int_{-\infty}^\infty f(x)\,dx = \int_{-\infty}^a f(x)\,dx + \int_a^\infty f(x)\,dx$$

Of particular interest is the integral $\int_1^\infty \frac{1}{x^p}\,dx$ which is convergent if $p > 1$ and divergent if $p \le 1$.

5.2 Type 2 Improper Integrals

We considered three cases involving discontinuities. If f is continuous on $[a, b)$ and is discontinuous at b, then

$$\int_a^b f(x)\,dx = \lim_{t\to b^-} f(x)\,dx,$$

if this limit exists (as a finite number). Similarly,

$$\int_a^b f(x)\,dx = \lim_{t\to a^+} f(x)\,dx.$$

If f has a discontinuity at c, where $a < c < b$, and both $\int_a^c f(x)\ dx$ and $\int_c^b f(x)\ dx$ are convergent, then we define

$$\int_a^b f(x)\ dx = \int_a^c f(x)\ dx + \int_c^b f(x)\ dx.$$

Hence, an integral over an interval where f has a discontinuity will diverge if the integral over either half diverges.