Math 240
Linear Algebra
October 3, 2018 - Exam 1

DIRECTIONS: Make sure to show all relevant work in a clear and complete fashion. **And be sure to show enough work to justify your answers!** You may do the problems in any order in your blue book, but be sure all parts of a given problem are located together. You may use your calculator for this exam. Good luck! There are 52 points on this exam.

Problem 1. (12 pts) The matrix

$$\mathbf{A} = \begin{bmatrix} 1 & -2 & 3 & 0 & -1 \\ 2 & -4 & 7 & -3 & 3 \\ 3 & -6 & 8 & 3 & -8 \end{bmatrix} \text{ is row equivalent to } \begin{bmatrix} 1 & -2 & 3 & 0 & -1 \\ 0 & 0 & 1 & -3 & 5 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

(a) Find $N(\mathbf{A})$ in parametric form.

(b) Find a basis for $C(\mathbf{A})$.

(c) Decide if $\begin{bmatrix} 4 \\ 10 \\ 10 \end{bmatrix}$ is in $C(\mathbf{A})$. Explain.

Problem 2. (8 pts) For what values of a, b, c, if any, is

$$\begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} \text{ a solution to } \begin{bmatrix} a & b & -3 \\ -2 & -b & c \\ a & 3 & -c \end{bmatrix} \mathbf{x} = \begin{bmatrix} -3 \\ -1 \\ -3 \end{bmatrix}?$$

Problem 3. (4 pts) Is the set $\left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \middle| x_4 = x_1 + x_2 \text{ and } x_3 = x_1 - x_2 \right\}$ a subspace of $\mathbb{R}^4$? If it is, find a basis for the subspace. If it is not, explain why not.

Problem 4. (15 pts) True or False? For each statement below, decide if it is true or false. If it is true, briefly explain why. If it is false, provide a specific example or cite a definition or theorem which shows the statement is false.

(a) If $\mathbf{A}$ is an $m \times n$ matrix with rank r, then $dim N(\mathbf{A}) = m - r$.

(b) If $\{\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}, \mathbf{v_4}\}$ is a set of linearly dependent set of vectors in $\mathbb{R}^3$ then $\{\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}\}$ must be a linearly independent set of vectors.

(c) Any line in $\mathbb{R}^2$ is a subspace of $\mathbb{R}^2$.

(d) If $\mathbf{u}$ and $\mathbf{w}$ are linear combinations of $\{\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}\}$ then $\mathbf{2w} - \mathbf{u}$ is in the span of $\{\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}, \mathbf{v_4}, \mathbf{v_5}, \mathbf{v_6}\}$.

(e) if $\mathbf{Ax} = \mathbf{b}$ has a solution for every $\mathbf{b}$ in a vector space V, then the columns of $\mathbf{A}$ form a basis for V.

Problem 5. (9 pts) Let A be an $n \times n$ matrix. Give 3 statements which are equivalent to saying that A is invertible. (State them carefully!)

Problem 6. (4 pts) Construct a 3×3 matrix A such that $N(A)$ is the set of vectors making up the z-axis and $C(A)$ is the $z = 0$ plane in $\mathbb{R}^3$ or explain why it can't be done.

CHALLENGE (2 pts) This will only be graded if you've attained at least 46 points above, so don't spend time on this unless you've got time to burn.

Let $\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_n}$ be a basis for $\mathbb{R}^n$. Let $\mathbf{b}$ be any vector in $\mathbb{R}^n$. Prove that $\mathbf{b}$ can be written uniquely (ie, there is only one way to do it) as a linear combination of $\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_n}$. Be sure to write in complete sentences.