Math 240 - Linear Algebra
November 24, 2019 - Final Exam

DIRECTIONS: Make sure to show all relevant work in a clear and complete fashion. You must show enough work to justify your answer! You may do the problems in any order in your blue book, but be sure to do all parts of a given problem are located together. There are 91 points on this exam.
Good luck!

Problem 1. (18 pts) Let $T : \mathbb{R}^5 \to \mathbb{R}^3$ be the linear transformation given by $T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} \right) = \begin{bmatrix} x_1 + 2x_3 \\ -x_4 \\ x_5 + 2x_3 + x_4 \end{bmatrix}$

   (a) Find the associated standard matrix of T.

   (b) Find $N(A)$.

   (c) Is T 1-1? Onto? Invertible? Justify your responses.

   (d) Find the matrix or product of matrices which represents $T : (\mathbb{R}^5, \alpha) \to (\mathbb{R}^3, \gamma)$ where

$$\alpha = \left\{ \begin{bmatrix} 5 \\ 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}, \begin{bmatrix} 4 \\ 5 \\ 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 3 \\ 4 \\ 5 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 4 \\ 5 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{bmatrix} \right\} \text{ and } \gamma = \left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix} \right\}.$$

Problem 2. (6 pts) Students are buying books for the new quarter. E buys the stats book, the discrete math book and the psychology book for $256. L buys books for herself and her friend, spending $319 for 2 stats books, one discrete math book, and one psychology book. M buys the psych book as well as the discrete math book for $193. Is there enough information to decide how much each book cost? Be sure to write a matrix equation to use in justifying your answer.

Problem 3. (10 pts) Consider the matrix A=
$$\begin{bmatrix} k & k-1 \\ k-1 & k \end{bmatrix}$$

where k is MY favorite real number.

   (a) For which values of k is this matrix invertible? Explain.

   (b) Find the eigenvectors and eigenvalues of A (in terms of k).

Problem 4. (21 pts) True or False? For each statement below, decide whether it is true or false. If it is true, briefly explain why. If it is false, cite a definition, theorem or calculation that shows the statement to be false.

   (a) If $T : V \to W$ is a linear transformation and $dim(V) = d, dim(W) = k$ and $dim(ImT) = c$, then $dim(KerT) = k - c$.

   (b) If $T : \mathbb{R}^3 \to \mathbb{R}^4$ is a linear transformation, and $dim(KerT) = 3$, then $T(v) = 0$ for all v.

   (c) If b is in $Span\{v_1, v_2, v_3\}$ then v_1 is in $Span\{b, v_2, v_3\}$.

   (d) The set $\{(0,0,1,-1),(2,3,2,-2)\}$ is a basis for the subspace of $\mathbb{R}^4$ given by $\{(x_1, x_2, x_3, x_4)|x_3+x_4 = 0\}$.

   (e) If $T : V \to W$ is a linear transformation, and $\{b_1, b_2, b_3, b_4\}$ is a basis for V, then $T(b_1), T(b_2), T(b_3)$ and $T(b_4)$ are linearly independent.

(f) If W is a 3 dimensional subspace of $\mathbb{R}^5$, then the orthogonal complement of the orthogonal complement is equal to the subspace W.

(g) The transformation $T : M_{n \times n} \to \mathbb{R}$ which is given by $T(A) = det(A)$ is a linear transformation.

**Problem 5.** (12 pts) Let $T : \mathbb{P}_3 \to M_{2 \times 2}$ be the linear transformation given by

$$T(a_0 + a_1 t + a_2 t^2 + a_3 t^3) = \begin{bmatrix} 2a_3 - 3a_2 + a_1 & a_1 + a_2 + a_3 \\ a_0 - 3a_2 - a_3 & a_0 - a_2 - a_3 \end{bmatrix}$$

(a) Compute $T(5t - 1 - 3t^3), T(1 + t + t^3)$ and $T(2t^3 - 1 - 2t)$. Are these vectors in $M_{2 \times 2}$ linearly independent? Explain.

(b) Is the set $W = \{p(t) \in \mathbb{P}_3 | p(0) = p(1)\}$ a subspace of $\mathbb{P}_3$? Justify your answer.

Problem 6. (8 pts) Consider the image shown here:
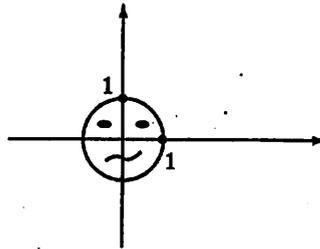
(a) Draw the image of the shape under the transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$, $T(\mathbf{v}) = \begin{bmatrix} 1 & 0 \\ 2 & \frac{1}{2} \end{bmatrix} \mathbf{v}$.

(b) Find the matrix A such that the image of the (original) shape under the transformation $T(\mathbf{v}) = A\mathbf{v}$ is as shown below or explain why it cannot be done.
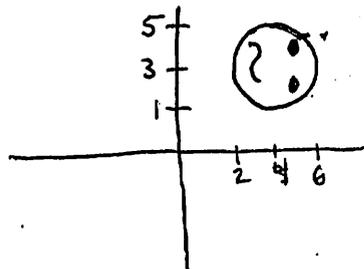
Problem 7. (4 pts) If B is some matrix such that $B\mathbf{x}$ is a multiple of the vector $\begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix}$ for every vector $\mathbf{x}$ in $\mathbb{R}^5$, then what, if anything, can you say about the rank of B? Explain.

Problem 8. (12 pts) Let V be the vector space of all 3 x 3 matrices whose only non-zero entries are on the diagonal and anti-diagonal. Define a linear transformation $T : V \to \mathbb{P}_2$ by

$$T\left(\begin{bmatrix} a_1 & 0 & b_1 \\ 0 & c_1 & 0 \\ a_2 & 0 & b_2 \end{bmatrix}\right) = (a_1 + 2c_1) - a_2 t + (a_2 + 2c_1 + b_2)t^2.$$

(a) Find a basis for $Ker(T)$. Be sure to use vectors in the appropriate vector space, and not coordinates.

(b) Find a basis for $Im(T)$. Be sure to use vectors in the appropriate vector space, and not coordinates.

(c) Prove or disprove: $H = \{\mathbf{v} \in V | \mathbf{v} \text{ is invertible}\}$ is a subspace of V.