Notes on Exam (Nov '19 Final)

Problem 1: a) $A = \begin{bmatrix} 1 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 2 & 1 & 1 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 1 & 1 \\ 0 & 0 & 0 & -1 & 0 \end{bmatrix}$ b) $N(A) = \text{Span}\left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1/2 \\ 0 \\ 1 \end{bmatrix} \right\}$

c) T is NOT 1-1 since $T\begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} = T\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ T IS ONTO since $\dim C(A) = 3$ so $C(A) =$ codomain

T is NOT invertible since it is not 1-1. (and since it's Matrix A is not even square!)

d) $T: (\mathbb{R}^5, \alpha) \xrightarrow{P_\alpha} (\mathbb{R}^5, \text{std}) \xrightarrow{A} (\mathbb{R}^3, \text{std}) \xrightarrow{P_\gamma^{-1}} (\mathbb{R}^3, \gamma)$

$$\frac{1}{18}\begin{bmatrix} -5 & 1 & 7 \\ 7 & -5 & 1 \\ 1 & 7 & -5 \end{bmatrix} \begin{bmatrix} 1 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 2 & 1 & 1 \end{bmatrix} \begin{bmatrix} 5 & 4 & 3 & 2 & 1 \\ 1 & 5 & 4 & 3 & 2 \\ 2 & 1 & 5 & 4 & 3 \\ 3 & 2 & 1 & 5 & 4 \\ 4 & 3 & 2 & 1 & 5 \end{bmatrix}$$

Problem 2: $\begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} S \\ D \\ P \end{bmatrix} = \begin{bmatrix} 256 \\ 319 \\ 193 \end{bmatrix}$ Since $\begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ there is a free variable, so NO – there are infinitely many possibilities.

Problem 3: a) $\det\begin{bmatrix} k & k-1 \\ k-1 & k \end{bmatrix} = 2k+1 = 0$ iff $k = -\frac{1}{2}$. So, invertible for all k EXCEPT $k = -\frac{1}{2}$.

b) char poly: $(k-\lambda)^2 - (k-1)^2 = 0$ roots are $\lambda = 1$, $\lambda = 2k-1$ (do it!).

$\lambda = 1$ $\begin{bmatrix} k-1 & k-1 \\ k-1 & k-1 \end{bmatrix}$ so e-vectors are $\text{Span}\left\{\begin{bmatrix} -1 \\ 1 \end{bmatrix}\right\}$ (if $k \neq 1$).

$\lambda = 2k-1$ $\begin{bmatrix} 1-k & k-1 \\ k-1 & 1-k \end{bmatrix}$ so e-vectors are $\text{Span}\left\{\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right\}$ (if $k \neq 1$).

Problem 4: a) False. Rank Thm says $\dim(\ker T) + \dim(\operatorname{Im} T) = \dim V$ so $\dim \ker T = d - c$.

b) True Rank Thm says $\dim \ker T + \dim \operatorname{Im} T = \dim V$ so, $\dim \operatorname{Im} T = 0$ here

c) False. Suppose $v_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $v_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, $v_3 = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$ and $b = \begin{bmatrix} 0 \\ 3 \end{bmatrix}$

d) False. The vectors are L.I, and they are in the subspace, but the subspace is dim 3: x_1, x_2 can be any value. So can x_4, then x_3 is determined. If $(1,0,0,0)$ is ~~added~~ ~~then~~ we would have a basis.

e) False. Consider $T: (\mathbb{R}^4, \text{std}) \to (\mathbb{R}^3 \text{std})$ given by $T(x) = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ -1 & -1 & -1 & -1 \end{bmatrix}$

f) True. This is a theorem.

g) False $\det(cA) = c^n \det A$ but if $c = 2$ and $n \geq 2$, then $2^n \det A \neq 2 \det A$.

<u>Problem 5</u> a) $T(5t-1-3t^3) = \begin{bmatrix} -1 & 2 \\ 2 & 2 \end{bmatrix}$ $T(1+t+t^3) = \begin{bmatrix} 3 & 2 \\ 0 & 0 \end{bmatrix}$ $T(2t^3-1-2t) = \begin{bmatrix} 2 & 0 \\ -3 & -3 \end{bmatrix}$

$c_1 \begin{bmatrix} -1 & 2 \\ 2 & 2 \end{bmatrix} + c_2 \begin{bmatrix} 3 & 2 \\ 0 & 0 \end{bmatrix} + c_3 \begin{bmatrix} 2 & 0 \\ -3 & -3 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \rightsquigarrow \left[\begin{array}{ccc|c} -1 & 3 & 2 & 0 \\ 2 & 2 & 0 & 0 \\ 2 & 0 & -3 & 0 \\ 2 & 0 & -3 & 0 \end{array}\right] \rightsquigarrow \left[\begin{array}{ccc|c} -1 & 3 & 2 & 0 \\ 0 & 8 & 4 & 0 \\ 0 & 0 & 16 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$ Yes! L.I.

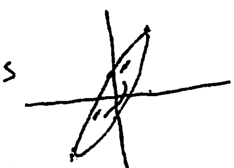
b). $W = \{p(t) \in P_3 \mid p(0) = p(1)\} = \{a_0 + a_1 t + a_2 t^2 + a_3 t^3 \mid a_0 = a_0 + a_1 + a_2 + a_3\}$

Since $0 = 0+0+0+0$ $\vec{0} \in W$.

If $a_0 = a_0 + a_1 + a_2 + a_3$ then $ca_0 = c(a_0 + a_1 + a_2 + a_3)$ and $ca_0 = ca_0 + ca_1 + ca_2 + ca_3$
So closed under scalar mult.

If $a_0 = a_0 + a_1 + a_2 + a_3$ and $b_0 = b_0 + b_1 + b_2 + b_3$ then
$(a_0 + b_0) = (a_0 + a_1 + a_2 + a_3) + (b_0 + b_1 + b_2 + b_3)$ and so $(a_0 + b_0) = a_0 + b_0 + a_1 + b_1 + a_2 + b_2 + a_3 + b_3$
so closed under addition.

Yes! A subspace.

<u>Problem 6</u> a) Since $T\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ and $T\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ \frac{1}{2} \end{pmatrix}$ Image is 

b). Can't be done! $T\begin{pmatrix} 0 \\ 0 \end{pmatrix} \neq \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ here so this is not linear.

<u>Problem 7</u>: We are told B is 3×5, and $C(B) = \text{Span}\left\{\begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix}\right\}$. So $\text{rk} B = 1$.

<u>Problem 8</u>: a) $\text{Ker}\, T = \left\{ \begin{bmatrix} a_1 & a & b_1 \\ 0 & c_1 & 0 \\ a_2 & 0 & b_2 \end{bmatrix} \middle| \begin{array}{l} a_1 + 2c_1 = 0 \\ -a_2 = 0 \\ a_2 + 2c_1 + b_2 = 0 \end{array} \right\}$. $\begin{array}{c} \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & -1 & 0 & 0 \\ 0 & 1 & 1 & 2 \end{bmatrix} \\ \begin{array}{cccc} a_1 & a_2 & b_2 & c_1 \end{array} \end{array} \rightsquigarrow \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 2 \end{bmatrix}$

Since c_1 & b_1 have no restrictions, $\dim \text{Ker}\, T = 2$,
basis is. $\left\{ \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} -2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{bmatrix} \right\}$

b) Note: $\dim V = 5$, $\dim \text{Ker}\, T = 2$ so $\dim \text{Im}\, T = 3$ by the rank thm.
Since ~~dim~~ $\text{Im}\, T = \dim \mathbb{P}_2$, a basis is $\{1, t, t^2\}$.
Another good choice is choosing Largest LI set in $T\left(\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\right), T\left(\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\right)$ etc..
so $\{1, 2 + 2t^2, -t + t^2\}$

c) This is <u>not</u> a subspace since $\vec{0}$ is not in H.