1. Use the definition of the definite integral to evaluate [1]

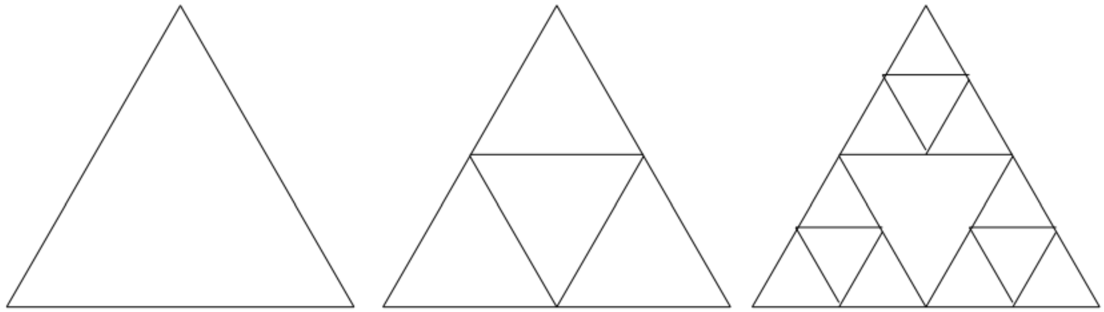
$$\int_{-1}^{1} (3x - x^2)\, dx.$$

SOLUTION: Find $\Delta x = \frac{2}{n}$, $x_i = -1 + \frac{2i}{n}$, and $f(x_i) = 3(-1 + \frac{2i}{n}) - (-1 + \frac{2i}{n})^2$.

$$\begin{aligned}
\lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x &= \lim_{n\to\infty} \sum_{i=1}^{n} \left[\left(3\left(-1 + \frac{2i}{n}\right) - \left(-1 + \frac{2i}{n}\right)^2 \right) \left(\frac{2}{n}\right) \right] \\
&= \lim_{n\to\infty} \sum_{i=1}^{n} \left[\left(-3 + \frac{6i}{n} - \left(1 - \frac{4i}{n} + \frac{4i^2}{n^2}\right) \right) \left(\frac{2}{n}\right) \right] \\
&= \lim_{n\to\infty} \sum_{i=1}^{n} \left[\left(-4 + \frac{10i}{n} - \frac{4i^2}{n^2} \right) \left(\frac{2}{n}\right) \right] \\
&= \lim_{n\to\infty} \sum_{i=1}^{n} \left[\left(-\frac{8}{n} + \frac{20i}{n^2} - \frac{8i^2}{n^3} \right) \right] \\
&= \lim_{n\to\infty} \left[-\frac{8}{n} \sum_{i=1}^{n} 1 + \frac{20}{n^2} \sum_{i=1}^{n} i - \frac{8}{n^3} \sum_{i=1}^{n} i^2 \right] \\
&= \lim_{n\to\infty} \left[-\frac{8n}{n} + \frac{20}{n^2}\left(\frac{n^2+n}{2}\right) - \frac{8}{n^3}\left(\frac{(n^2+n)(2n+1)}{6}\right) \right] \\
&= \lim_{n\to\infty} \left[-8 + \frac{20n^2}{2n^2} + \frac{20n}{2n^2} - \frac{8}{6n^3}(2n^3 + 3n^2 + n) \right] \\
&= \lim_{n\to\infty} \left[-8 + 10 + \frac{10}{n} - \frac{16}{6} - \frac{24}{6n} - \frac{8}{6n^2} \right] \\
&= -8 + 10 - \frac{16}{6} \\
&= -\frac{2}{3}
\end{aligned}$$

2. The *Sierpinski Triangle* is constructed in the following way: Start with an equilateral triangle of side length $s/2$ and area $A/4$. Perform the center-removing operation on each of these three triangles. The result will be 9 triangles, and so on.... See the figure.

[1]
$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2} \quad \sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6} \quad \sum_{i=1}^{n} i^3 = \frac{n^2(n+1)^2}{4}$$



(a) Use geometric series arguments to show that the total area removed is A, i.e. the area of the Sierpinski triangle is 0.

(b) Show that the length of the boundary of the Sierpinski triangle is infinite.

SOLUTION:

(a) We can note that the sequence representing the remaining area can be written as $\{A, \frac{3A}{4}, \frac{9A}{16}, ...\}$ OR the sequence of removed area is $\{\frac{A}{4}, \frac{3A}{16}, \frac{9A}{64}, ...\}$. As a geometric series this can be written as

$$\sum_{n=0}^{\infty} \frac{A}{4}\left(\frac{3}{4}\right)^n$$

which converges to $\left|\frac{A/4}{1-3/4}\right| = A$ for $|r| = 3/4 < 1$.

(b) The length of the boundary can be written as the sequence $\{3s, \frac{9s}{2}, \frac{27s}{4}, ...\}$

$$\sum_{n=1}^{\infty} \frac{3^n}{2^{n-1}} s$$

In this case, the $|r| = 3/2 > 1$ implies that the series diverges and thus the boundary is inifite.

3. Use integration by substitution to show that $\int f(\sqrt{x})\ dx = \int 2uf(u)\ du$. Calculate $\int \sin(\sqrt{x})\ dx$.

SOLUTION: Let $u = \sqrt{x}$ and $du = \frac{1}{2\sqrt{x}}dx$. We can manipulate the second equation so we have $dx = 2\sqrt{x}\ du = 2u\ du$. Now, our integral becomes

$$\int f(u)2u\ du.$$

Let's put this to the test with the function $f(x) = \sin(\sqrt{x})$.

Use IBP with $t = 2u \implies dt = 2\ du$ and $ds = \sin(u)\ du \implies s = -\cos(u)$.

$$\begin{aligned}\int \sin(u)2u\ du &= -2u\cos(u) + 2\int \cos(u)\ du \\ &= 2\sin(u) - 2u\cos(u) + C\end{aligned}$$

4. A mathematical operation that takes a function as input and produces another function as output is sometimes referred to as a *transform.* An important example is the *transform* $\mathcal{L}$, defined by

$$\mathcal{L}[f(x)] = \int_0^\infty e^{-px} f(x)\ dx = F(p).$$

Notice that $\mathcal{L}$ turns a function of x into a function of the parameter p. Find the Laplace transform of the function $f(x) = 1$, $g(x) = x$, and $h(x) = e^{5x}$. (Assume in the first two cases that $p > 0$ and in the third case $p > 5$.)

SOLUTION:

For $f(x) = 1$ we have the integral

$$\begin{aligned}\int_0^\infty e^{-px}\ dx &= \frac{-1}{p}e^{-px}\Big|_0^\infty \\ &= \lim_{b\to\infty}\left(\frac{1}{pe^{pb}}\right) + \frac{1}{p} \\ &= \frac{1}{p}\end{aligned}$$

For $g(x) = x$ we have the integral

$$\int_0^\infty e^{-px}x\ dx.$$

Here we must turn to our trusty IBP method with $u = x$, $du = dx$, $dV = e^{-px}dx$, and $V = \frac{-1}{p}e^{-px}$.

$$\begin{aligned}\int_0^\infty e^{-px}x\ dx &= \frac{-x}{p}e^{-px}\Big|_0^\infty + \int_0^\infty \frac{1}{p}e^{-px}dx \\ &= (0-0) + \frac{-1}{p^2}e^{-px}\Big|_0^\infty \\ &= \frac{1}{p^2}\end{aligned}$$

These first two solutions hold since $p > 0$.

Finally, for $h(x) = e^{5x}$ we must solve the following integral:

$$\begin{aligned}\int_0^\infty e^{-px}e^{5x}\,dx &= \int_0^\infty e^{(5-p)x}\,dx \\ &= \frac{1}{5-p}\,e^{(5-p)x}\Big|_0^\infty \\ &= \frac{-1}{5-p} \\ &= \frac{1}{p-5}\end{aligned}$$

Luckily, we are told to assume that $p > 5$ so this fraction is defined.

5. Find the limit:

$$\lim_{h\to 0}\frac{\int_1^{1+h}\sqrt{x^5+8}dx}{h}$$

SOLUTION: Use L'Hospital's rule and FTC:

$$\begin{aligned}\lim_{h\to 0}\frac{\frac{d}{dh}\int_1^{1+h}\sqrt{x^5+8}dx}{\frac{d}{dh}h} &= \lim_{h\to 0}\frac{\sqrt{(1+h)^5+8}}{1} \\ &= \frac{\sqrt{9}}{1} \\ &= 3\end{aligned}$$

6. Starting with

$$f(x) = \sum_{n=0}^{\infty} x^n,$$

find the sum of the series:

(a)

$$\sum_{n=1}^{\infty} nx^{n-1}, \quad |x| < 1$$

(b)

$$\sum_{n=1}^{\infty}\frac{n}{2^n}$$

(c)

$$\sum_{n=2}^{\infty} n(n-1)x^n, \quad |x| < 1$$

(d)

$$\sum_{n=2}^{\infty} \frac{n^2 - n}{2^n}$$

(e)

$$\sum_{n=1}^{\infty} \frac{n^2}{2^n}$$

SOLUTION:

(a) We see that this looks like the first derivative of $\sum_{n=1}^{\infty} x^n$.

$$\frac{d}{dx} \sum_{n=1}^{\infty} x^n = \frac{d}{dx} \frac{1}{1-x}$$

Thus,

$$\sum_{n=1}^{\infty} nx^{n-1} = \frac{1}{(1-x)^2}$$

(b) We apply what we found in the first part as this looks like the first derivative of $\sum_{n=1}^{\infty} x^n$ with $x = \frac{1}{2}$.

$$\sum_{n=1}^{\infty} n \cdot \left(\frac{1}{2}\right)^n \implies x = 1/2$$

$$\frac{1}{(1-\frac{1}{2})^2} = 4$$

(c) Similarly, this looks like the second derivative of $\sum_{n=1}^{\infty} x^n$. So we differentiate $\frac{1}{1-x}$ to find what the sum converges to.

$$\frac{d}{dx} \frac{1}{(1-x)^2} = \frac{2}{(1-x)^3}$$

(d) Similar to part b, we apply part c with $x = \frac{1}{2}$, so

$$\frac{2}{(1-\frac{1}{2})^3} = \frac{2}{1/8} = 16$$

(e) This is the trickiest part. If we examine what we have previously solved, it is apparent that this is the subtraction of the two sums above.

$$\sum_{n=2}^{\infty} \frac{n^2 - n}{2^n} - \sum_{n=1}^{\infty} \frac{n}{2^n} = 16 - 4 = 12$$

7. Use either a direct comparison or the limit comparison test to decide the convergence of divergence of each of the following:

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{1+n^5}} \quad \sum_{n=3}^{\infty} \frac{n}{(n+3)^2} \quad \sum_{n=1}^{\infty} \frac{1}{2n^2-n} \quad \sum_{n=1}^{\infty} \frac{5\sin^2 n}{n\sqrt{n}}$$

SOLUTION:

For the first sum we can perform a direct comparison test. We know that

$$1+n^5 > n^5 \implies \frac{1}{\sqrt{1+n^5}} < \frac{1}{\sqrt{n^5}} = \frac{1}{n^{5/2}}.$$

Since $\sum_1^{\infty} \frac{1}{n^{5/2}}$ converges because $p > 1$, we also know that $\sum_{n=1}^{\infty} \frac{1}{\sqrt{1+n^5}}$ will converge.

For the second sum we can also use a comparison test. If we expand the bottom of the fraction we can note that

$$n^2 < n^2 + 6n + 9$$

and so it follows that

$$\frac{n}{n^2} > \frac{n}{n^2+6n+9}.$$

Since $\sum_{n=3}^{\infty} \frac{1}{n}$ diverges, we know that $\sum_{n=3}^{\infty} \frac{n}{(n+3)^2}$.

For the third sum we can use a comparison test with $\frac{1}{n^2}$. Since

$$2n^2 - n \geq n^2, \;\; \forall n \geq 1,$$

we know that

$$\frac{1}{2n^2-n} \leq \frac{1}{n^2}.$$

Since $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges, we know that $\sum_{n=1}^{\infty} \frac{1}{2n^2-n}$ also converges.

We could also use the limit comparison test here with $a_n = \frac{1}{n^2}$ and $b_n = \frac{1}{2n^2-n}$. Then

$$\lim_{n\to\infty} \frac{1/n^2}{\frac{1}{2n^2-n}} = \lim_{n\to\infty} \frac{2n^2}{n^2} = 2.$$

Since this is finite and positive we know that both sequences converge because a_n converges.

Finally, for the last sum we know that $0 < \sin^2(n) < 1$ or $0 < 5\sin^2(n) < 5$. Thus,

$$\frac{5\sin^2(n)}{n\sqrt{n}} \leq \frac{5}{n\sqrt{n}} \approx \frac{1}{n\sqrt{n}}, \text{ for large n.}$$

Now we have $\sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$ which diverges since the exponent is greater than one. Thus, this sum also converges.

8. When a plane region $\mathcal{R}$ is revolved around an axis A, a solid of revolution S is generated. Let CM be the center of mass of $\mathcal{R}$, and $\bar{C}$ be the circle generated when CM is revolved around the axis A. The Theorem of Pappus says that:

$$\text{Volume of } S = (\text{Circumference of } \bar{C})\cdot(\text{Area of } \mathcal{R}).$$

Suppose $\mathcal{R}$ is the region between the graph of $y = f(x)$ and the x-axis from $x = a$ to $x = b$, where $0 < a < b$ and $f(x) > 0$, Show that the Theorem of Pappus is true for the solid of revolution S generated by revolving $\mathcal{R}$ about the y-axis. [HINT Use the method of shells.]

Recall the torus from problem 10 of the second exam problem set. Use the Theorem of Pappus to find its volume. How does this answer compare with your result using washers or shells?

SOLUTION:

Using the method of shells we have

$$V = \int_a^b 2\pi x f(x)\, dx$$

We want to arrive at $V = 2\pi \overline{x} A$ so we can multiply by 1 in a clever way.

$$\begin{aligned} V &= \int_a^b 2\pi x f(x) \cdot \frac{\int_a^b f(x)\, dx}{\int_a^b f(x)\, dx}\, dx \\ &= 2\pi \frac{\int_a^b x f(x)\, dx}{\int_a^b f(x)\, dx} \int_a^b f(x)\, dx \\ &= 2\pi \overline{x} A \end{aligned}$$

We have left the application of this theorem as an exercise to the reader as we did not solve a problem 10 on a previous exam review involving a torus. It should be fairly straight forward as a torus is merely the rotation of circle around the y-axis thus finding the center of mass and the area is trivial.

9. Consider a thin, homogeneous, rigid rod of length L cm and density d g/cm, aligned on the x-axis with one end fixed at the origin.

Now suppose the rod is rotating around the y-axis with angular velocity w rev/sec, like a helicopter rotor. Set up and calculate the integral to find the total kinetic energy of the rotating rod.

THE PROCEDURE: Go back to basic principles. Dissect the rod into small pieces, figure out the kinetic energy of each small bit, and then add all the contributions to the total kinetic energy with an appropriate integral. The kinetic energy of a point mass m is $\frac{1}{2}mv^2$ where v is the *linear velocity*, measured in units of cm/sec. The linear velocity of a point on the rotating rod r units away from the origin is $2\pi r\omega$.

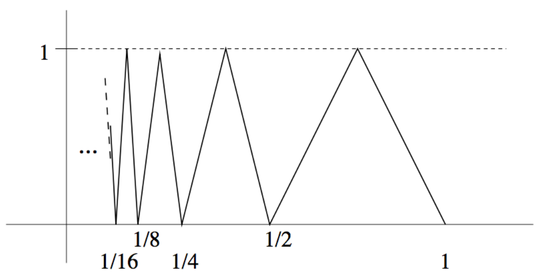
SOLUTION:

$$\begin{aligned}\int_0^L \frac{1}{2}mv^2\,dr &= \int_0^L \frac{1}{2}m(2\pi r\omega)^2\,dr\\ &= 2m\pi^2\omega^2\frac{r^3}{3}\Big|_0^L\\ &= \frac{2}{3}m\pi^2\omega^2L^3\end{aligned}$$

10. Evaluate the definite integral

$$\int_0^1 f(x)dx$$

where f is the function whose graph is shown below



SOLUTION:

Adding the area under the curve (for each triangle) we have:

$$\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)(1)+\left(\frac{1}{2}\right)\left(\frac{1}{4}\right)(1)+\left(\frac{1}{2}\right)\left(\frac{1}{8}\right)(1)+\left(\frac{1}{2}\right)\left(\frac{1}{16}\right)(1)+...$$

We can rewrite this as the geometric series

$$\sum_{n=1}^{\infty}\frac{1}{2}\left(\frac{1}{2}\right)^n.$$

This will converge to

$$\frac{a}{1-r}=\frac{1/2}{1-1/2}=1$$

for $|r|<1$.

11. Here is a "proof" that 1=0. Your job is to find the bug.

On the one hand,

$$\begin{aligned}\frac{2}{\pi}\int_0^{\pi}\cos^2\theta\ d\theta &= \frac{2}{\pi}\int_0^{\pi}\frac{1+\cos 2\theta}{2}\ d\theta\\ &= \frac{2}{\pi}(\frac{1}{2}\theta+\frac{1}{4}\sin 2\theta)|_0^{\pi}\\ &= \frac{2}{\pi}\frac{\pi}{2}\\ &= 1\end{aligned}$$

On the other hand, let $u=\sin\theta$. Then $\cos\theta=\sqrt{1-\sin^2\theta}=\sqrt{1-u^2}$ and $du=\cos\theta d\theta$, so

$$\begin{aligned}\frac{2}{\pi}\int_0^{\pi}\cos^2\theta d\theta &= \frac{2}{\pi}\int_0^{\pi}(\cos\theta)(\cos\theta)d\theta\\ &= \frac{2}{\pi}\int_{\sin 0}^{\sin\pi}\sqrt{1-u^2}du\\ &= 0\end{aligned}$$

SOLUTION: The first solution is correct! In the second part, there is an algebra error that occurs in the last two lines. Since $\cos(\theta)=\sqrt{1-u^2}$, $\cos^2(\theta)$ should be equal to $1-u^2$. Furthermore, our new bounds both become 0 – making this integral seem to integrate to zero. This is where the major error occurs – to use substitution our support must be 1-1 and this is not the case here since both 0 and π take our function to the same value. Thus, the second method is invalid and the correct answer is 1.

12. (a) Use the Taylor's Theorem to find the Maclaurin series for $f(x)=e^x$.

(b) Use part (a) to find the Maclaurin series for e^{-x^3}. What is the interval of convergence of this series? For what x does it converge absolutely?

(c) Calculate

$$\int_0^{0.2} e^{-x^3}\ dx$$

to 4-decimal place accuracy. Explain how you know that your answer is sufficiently accurate.

SOLUTION:

(a) We need to use the formula

$$f(x)=f(0)+xf'(0)+\frac{x^2}{2!}f''(0)+...$$

Thus, the Maclaurin series for $f(x)=e^x$ is

$$1+x+\frac{x^2}{2}+\frac{x^3}{6}+...=\sum_{n=0}^{\infty}\frac{x^n}{n!}$$

(b) This becomes quite simple if we apply what we found from part (a) with the substitution $u = -x^3$. Now we have $f(u) = e^u$ and the solution we are looking for is:

$$\begin{aligned} \sum_{n=0}^{\infty} \frac{u^n}{n!} &= \sum_{n=0}^{\infty} \frac{(-3x)^n}{n!} \\ &= \sum_{n=0}^{\infty} (-1)^n \frac{x^{3n}}{n!} \end{aligned}$$

(c)

$$\begin{aligned} \int_0^{0.2} e^{-x^3}\, dx &= \int_0^{0.2} \sum_{n=0}^{\infty} (-1)^n \frac{x^{3n}}{n!} \\ &= \int_0^{0.2} \left(1 - x^3 + \frac{x^6}{2} - \frac{x^9}{6} + ...\right) dx \\ &= \left(x - \frac{x^4}{4} + \frac{x^7}{14} - \frac{x^{10}}{60} + ...\right)\Big|_0^{0.2} \\ &= .1996 \end{aligned}$$

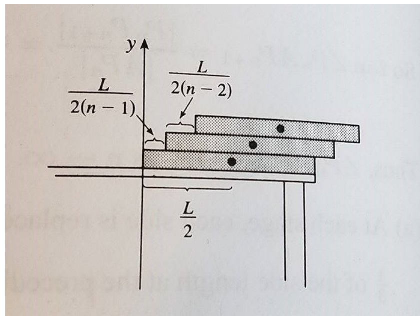
Note that we only needed to use the first two terms to get an answer accurate to four decimal places because the next term is on the order of 10^-7 – far too small to affect our answer.

13. Suppose you have a large supply of books, all the same size, and you stack them at the edge of a table, with each book extending farther beyond the edge of the table than the book beneath it: the top book extends half its length beyond the second book, the second book extends a quarter of its length beyond the third, the third extends one sixth of its length beyond the fourth, and so on.

    (a) Show, by considering the center of mass of the stack, that it is possible to stack the books so that the top book extends entirely beyond the table. Work from the top down. First find the center of mass of two books. Then use this result to find the center of mass when a third book has been added to the bottom of the stack. How many books does it take before the top book extends completely beyond the edge of the table?

    (b) By considering the harmonic series, show that the top book can extend *any distance at all* beyond the edge of the table if the stack is high enough.

SOLUTION:



(a) Let the length of each book be L and the mass m. We want to show that the center of mass lies above the table – or that $\overline{x} < L$. The coordinates of the centers of mass for the books are

$$x_1 = \frac{L}{2},\ \ x_2 = \frac{L}{2(n-1)} + \frac{L}{2},\ \ x_3 = \frac{L}{2(n-1)} + \frac{L}{2} + \frac{L}{2(n-2)}, \ldots$$

$$\begin{aligned}
\overline{x} &= \frac{mx_1 + mx_2 + \ldots mx_n}{mn} = \frac{x_1 + x_2 + \ldots + x_n}{n} \\
&= \frac{1}{n}\left[\frac{L}{2} + \left(\frac{L}{2(n-1)} + \frac{L}{2}\right) + \ldots + \left(\frac{L}{2(n-1)} + \frac{L}{2(n-2)} + \ldots + \frac{L}{4} + \frac{L}{2} + \frac{L}{2}\right)\right] \\
&= \frac{L}{n}\left[\frac{n-1}{2(n-1)} + \frac{n-2}{2(n-2)} + \ldots + \frac{2}{4} + \frac{1}{2} + \frac{n}{2}\right] \\
&= \frac{L}{n}\left[(n-1)\frac{1}{2} + \frac{n}{2}\right] \\
&= \frac{2n-1}{2n}L < L
\end{aligned}$$

Thus, no matter how many books are added, the center of mass lies above the table.

(b) We can write a sequence for the distance that the nth book extends from the last. The top book extends 1/2 the distance of the one below it and then 1/4 and then 1/6. So we can generalize the sequence to $\frac{1}{2n}$ where n is the nth book from the top. If we think about the series,

$$\sum_{n=1}^{\infty} \frac{1}{2n}$$

diverges. Thus, we can take n large enough so the top book is completely off the end of the table.