Math 240
Linear Algebra
October 12, 2022 - Exam 1

DIRECTIONS: Make sure to show all relevant work in a clear and complete fashion.And be sure to show enough work to justify your answers! You may do the problems in any order in your blue book, but be sure all parts of a given problem are located together. Good luck! There are 63 points on this exam.

Problem 1. (10 pts) The augmented matrix

$$\left[\begin{array}{cccc|c} 1 & -2 & 3 & 0 & -1 \\ 2 & -4 & 7 & -3 & 3 \\ 3 & -6 & 8 & 3 & -8 \end{array}\right] \text{ is row equivalent to } \left[\begin{array}{cccc|c} 1 & -2 & 3 & 0 & -1 \\ 0 & 0 & 1 & -3 & 5 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right].$$

(a) Find the set of solutions, in parametric form, or explain why there are no solutions.

(b) Find a basis for $C(\mathbf{A})$, where $\mathbf{A}$ is the NON-AUGMENTED portion of the matrix above.

Problem 2. (10 pts) Let

$$\mathbf{A} = \begin{bmatrix} 1 & 3 & 2 & 2 \\ 2 & 7 & 6 & 8 \\ 3 & 9 & 6 & 7 \end{bmatrix}.$$

(a) Find $\mathbf{b}$ such that $\mathbf{Ax} = \mathbf{b}$ has no solutions or explain why there are none.

(b) Are the columns of the matrix $\mathbf{A}$ linearly independent or not? Explain.

Problem 3. (5 pts) Let $\mathbf{A}$ be the matrix

$$\begin{bmatrix} 1 & 2 & 3 \\ 1 & 1 & -2 \\ -1 & 0 & k \end{bmatrix}.$$

Find all values of k such that the null space of $\mathbf{A}$ has dimension 1, or explain why it can't be done.

Problem 4. (6 pts)

(a) Give an example of a subspace of $Span\left\{\begin{bmatrix} 4 \\ 4 \\ 4 \\ 1 \\ 4 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 4 \\ 6 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 9 \\ 3 \\ 0 \\ 3 \end{bmatrix}\right\}$. (Make sure your vector space is non-trivial, that is, that it has more than one vector in it.)

(b) Verify that your example above really is a subspace.

Problem 5. (5 pts) Consider the vectors $\left\{\begin{bmatrix} 2 & -4 & 2 \\ -4 & 8 & -4 \end{bmatrix}, \begin{bmatrix} 2 & -2 & -1 \\ 0 & 4 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & -2 \\ k & 0 & k \end{bmatrix}\right\}$ in $M_{2\times 3}$. For which values of k will these three vectors be linearly independent? Justify your answer.

Problem 6. (12 pts) Let $A = \begin{bmatrix} 1 & 1 & 3 & 1 \\ 2 & 2 & a & 1 \\ 0 & 0 & 4 & b \\ 0 & 0 & 2 & 6 \end{bmatrix}.$

(a) For which pairs of values a and b, if any, will $\mathbf{A}$ be invertible? Explain.

(b) For which pairs of values a and b,if any, will $\mathbf{A}x = \mathbf{0}$ have exactly a line's worth of solutions? Explain.

(c) Give two equivalent statements to saying that a matrix $\mathbf{A}$ is invertible which are different than the reasoning you used in part (a).

Problem 7. (15 pts) True or False? For each statement below, decide whether it is true or false. If it is true, briefly explain why. If it is false, cite a definition, theorem or calculation that shows the statement to be false.

(a) If $\mathbf{b}$ is in $Span\{\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}\}$ then $\mathbf{v_3}$ is in $Span\{\mathbf{v_1}, \mathbf{v_2}, \mathbf{b}\}$

(b) For 2×2 matrices $\mathbf{B}, \mathbf{C}$, and $\mathbf{D}$ if $\mathbf{BC} = \mathbf{BD}$, then $\mathbf{C} = \mathbf{D}$.

(c) If $\mathbf{A}$ is an 203×97 matrix and $rank(\mathbf{A}) = 97$ then $\mathbf{Ax} = \mathbf{0}$ has no solutions.

(d) If the set $\{\mathbf{b_1}, \mathbf{b_2}, \ldots \mathbf{b_5}\}$ is a basis for $\mathbb{R}^5$, then the matrix $\begin{bmatrix} \mathbf{b_1} & \mathbf{b_2} & \mathbf{b_3} & \mathbf{b_4} & \mathbf{b_5} \end{bmatrix}$ whose columns are the $\mathbf{b_i}$ is invertible.

(e) If the columns of a matrix $\mathbf{A}$ are linearly independent, then the columns of $\mathbf{A}^T$ are linearly independent too.

CHALLENGE: (2 pts) This will only be graded if you've attained at least 52 points above, so don't spend time on this unless you've finished the rest of the exam.

Prove that if V is a vector space and $\{\mathbf{b_1}, \ldots \mathbf{b_n}\}$ is a basis for V then any vector in $Span\{\mathbf{b_1}, \ldots \mathbf{b_n}\}$ is a UNIQUE linear combination of those vectors. Be sure to write in complete sentences, and be sure your work applies to any vector space V (ie, your proof should NOT use matrices).