Remarks on Linear Algebra Review (Nov '18 final exam)

Problem 1: $\begin{bmatrix} 1&1&1&1\\1&2&3&3\\2&2&2&3\end{bmatrix} \rightsquigarrow \begin{bmatrix} 1&1&1&1\\0&1&2&2\\0&0&0&1\end{bmatrix}$

a) basis $N(A) = \left\{ \begin{bmatrix}1\\-2\\1\\0\end{bmatrix} \right\}$

b) $N(A)^{\perp} = N([1\ -2\ 1\ 0]) = \text{Span}\left\{ \begin{bmatrix}2\\1\\0\\0\end{bmatrix}, \begin{bmatrix}-1\\0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\0\\1\end{bmatrix} \right\}$

c) sol set: $\left\{ \begin{bmatrix}-5\\11\\0\\-5\end{bmatrix} + t\begin{bmatrix}1\\-2\\1\\0\end{bmatrix} \,\middle|\, t \in \mathbb{R} \right\}$

Problem 2: $A^TA = \begin{bmatrix} k^2&k&k\\k&2&1\\k&1&1\end{bmatrix} \xrightarrow[k\neq 0]{} \begin{bmatrix} k^2&k&k\\0&k&0\\0&0&0\end{bmatrix}$

a) never invertible since rref $\neq I$

b) none since $k \neq 0 \Rightarrow 2$ pivots

$k = 0 \Rightarrow 2$ pivots

So $\dim N(A^TA) = 1$

Problem 3: Since $\mathcal{B} = \left\{ \begin{bmatrix}2\\2\end{bmatrix}, \begin{bmatrix}3\\-5\end{bmatrix} \right\}$ is a basis for $\mathbb{R}^2$, this <u>can</u> be done.

Find: $\begin{bmatrix}1\\0\end{bmatrix} = \begin{bmatrix}5/16\\1/8\end{bmatrix}_{\mathcal{B}}$ $\begin{bmatrix}0\\1\end{bmatrix} = \begin{bmatrix}3/16\\-1/8\end{bmatrix}_{\mathcal{B}}$ so $T\begin{bmatrix}1\\0\end{bmatrix} = \begin{bmatrix}9/8\\2\end{bmatrix}$, $T\begin{bmatrix}0\\1\end{bmatrix} = \begin{bmatrix}7/8\\0\end{bmatrix} \Rightarrow T(\vec{x}) = \begin{bmatrix}9/8 & 7/8\\2 & 0\end{bmatrix}\vec{x}$

or Since $\mathbb{R}^2_{,\mathcal{B}} \xrightarrow{T} \mathbb{R}^2_{,\text{std}}$ is given by $A = \begin{bmatrix}4&-1\\4&6\end{bmatrix}$, compute $AP_{\mathcal{B}}^{-1}$ where

$P_{\mathcal{B}} = \begin{bmatrix}2&3\\2&-5\end{bmatrix}$, and $P_{\mathcal{B}}^{-1}: \mathbb{R}_{2,\text{std}} \to \mathbb{R}_{2,\mathcal{B}}$ is $\frac{-1}{16}\begin{bmatrix}-5&-3\\-2&2\end{bmatrix}$.

Problem 4: a) basis $\operatorname{Im} T = \left\{ \begin{bmatrix}0&0&1\\1&0&0\end{bmatrix}, \begin{bmatrix}-4&4&0\\0&1&2\end{bmatrix}, \begin{bmatrix}2&0&0\\0&0&-1\end{bmatrix}, \begin{bmatrix}0&0&-1\\1&0&0\end{bmatrix} \right\}$

b) $M_{3\times 2,\alpha} \xrightarrow{P_\alpha} M_{3\times 2,\text{std}} \xrightarrow{T} M_{2\times 3,\text{std}} \xrightarrow{P_{\mathcal{B}}^{-1}} M_{2\times 3,\mathcal{B}}$

$$\begin{bmatrix} 2&2&1&0&-6&4\\1&1&-4&7&7&0\\1&0&0&0&3&3\\0&2&-2&1&-1&-1\\4&0&8&3&3&0\\0&2&0&0&2&2\end{bmatrix}^{-1} \begin{bmatrix} 0&-4&2&0&0&0\\0&4&0&0&0&0\\1&0&0&0&-1&0\\1&0&0&0&1&1\\0&1&0&0&0&0\\0&2&-1&0&0&0\end{bmatrix} \begin{bmatrix} 2&0&5&1&2&-1\\0&4&1&1&0&2\\-1&6&1&1&0&0\\3&-2&0&3&-4&-7\\0&0&3&0&0&0\\0&0&-2&0&4&0\end{bmatrix}$$

Problem 5: Solve $\begin{bmatrix}1&2&0&0\\0&0&1&2\\3&4&0&0\\0&0&3&4\end{bmatrix}\begin{bmatrix}a_{11}\\a_{12}\\a_{21}\\a_{22}\end{bmatrix} = \begin{bmatrix}3\\6\\-6\\-8\end{bmatrix}$ or use $A = PDP^{-1}$

$= -\frac{1}{2}\begin{bmatrix}1&3\\2&4\end{bmatrix}\begin{bmatrix}3&0\\0&-2\end{bmatrix}\begin{bmatrix}4&-3\\-2&1\end{bmatrix}$

to get $\begin{bmatrix}-12 & 15/2\\-20 & 13\end{bmatrix}$

Problem 6

a) True - check!

b) False eg $\begin{bmatrix}1&0\\0&0\end{bmatrix}$

c) False $\begin{bmatrix}1&0\\0&1\\0&0\end{bmatrix}$

d) False $\begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}^{-1} = \begin{bmatrix} 1/2 & 0 \\ 0 & 1/3 \end{bmatrix}$

e) True A invertible $\Rightarrow A$ has pivot in each col $\Rightarrow A^T$ has pivot in each row $\Rightarrow A^T$ invertible

f) False $0+0-1 \neq 0$ so $\vec{0} \notin S$

g) False $\begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$ $v_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ $v_3 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ But $3v - w = \begin{bmatrix} 3 \\ -1 \end{bmatrix}$ and $A\begin{bmatrix} 3 \\ -1 \end{bmatrix} \neq \begin{bmatrix} 6 \\ -3 \end{bmatrix}$

not scalar mult.

h) True: $T(\vec{x}) = \begin{bmatrix} 2 & 1 \\ -2 & 4 \\ 1 & 6 \end{bmatrix} \vec{x}$ and $A = \begin{bmatrix} 2 & 1 \\ -2 & 4 \\ 1 & 6 \end{bmatrix}$ has 2 pivots $\Rightarrow N(A) = \{\vec{0}\}$.

i) False $\dim \ker T + \dim \operatorname{Im} T = \underset{n}{\underset{\shortparallel}{\dim V}}$

Problem 7: $\begin{bmatrix} | & | & | \\ T(1) & T(t) & T(t^2) \\ | & | & | \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 2 \\ 0 & 0 & 3 \end{bmatrix}$ 3 distinct e-values, so yes, $\exists$ e-vectors.

Problem 9: Let $T: V \to W$ be a 1-1 linear transformation, with $v_1, \dots v_k$ lin indep in V. Then $T(v_1), \dots T(v_k)$ are lin indep.

Pf: Since T is 1-1, $T(\vec{w}) = \vec{0}$ only if $\vec{w} = \vec{0}$. Since $\{v_1, \dots v_k\}$ is a lin indep set, none of these $v_i = 0$, so none of $T(v_i) = \vec{0}$. Consider $c_1 T(v_1) + \dots + c_k T(v_k) = \vec{0}$. By linearity, we rewrite the left hand side to get $T(c_1 v_1 + \dots + c_k v_k) = \vec{0}$. Again, since 1-1, this means $c_1 v_1 + \dots + c_k v_k = \vec{0}$. Since $\{v_1, \dots v_k\}$ is lin indep, $\Rightarrow c_1 = \dots = c_k = 0$. Thus, $\{T(v_1), \dots, T(v_k)\}$ is lin indep.