Linear Algebra Review (Oct 18 exam)

Problem 1: a) $N(A) = \left\{ x_2 \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -9 \\ 0 \\ 3 \\ 1 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} 16 \\ 0 \\ -5 \\ 0 \\ 1 \end{bmatrix} \;\middle|\; x_2, x_4, x_5 \in \mathbb{R} \right\}$

b) basis for $C(A) = \left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 3 \\ 7 \\ 8 \end{bmatrix} \right\}$

c) $\left[\begin{array}{cc|c} 1 & 3 & 4 \\ 2 & 7 & 10 \\ 3 & 8 & 10 \end{array}\right] \rightsquigarrow \left[\begin{array}{cc|c} 1 & 3 & 4 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{array}\right]$ has a solution, so yes.

Problem 2: Convert to $\left[\begin{array}{ccc|c} 1 & -1 & 0 & 3 \\ 0 & 1 & 2 & 1 \\ 1 & 0 & -2 & 0 \end{array}\right] \rightsquigarrow \left[\begin{array}{ccc|c} 1 & -1 & 0 & 3 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 4 & 4 \end{array}\right] \Rightarrow$ $c=1$, $b=-1$, $a=2$ is unique choice.

Problem 3: Note $\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$ satisfies the conditions.

Note: $\begin{bmatrix} x_1 \\ x_2 \\ x_1-x_2 \\ x_1+x_2 \end{bmatrix} + \begin{bmatrix} y_1 \\ y_2 \\ y_1-y_2 \\ y_1+y_2 \end{bmatrix} = \begin{bmatrix} x_1+y_1 \\ x_2+y_2 \\ (x_1-x_2)+(y_1-y_2) \\ (x_1+x_2)+(y_1+y_2) \end{bmatrix} = \begin{bmatrix} x_1+y_1 \\ x_2+y_2 \\ (x_1+y_1)-(x_2+y_2) \\ (x_1+y_1)+(x_2+y_2) \end{bmatrix}$ so closed under ~~the~~ addition.

Note: $c\begin{bmatrix} x_1 \\ x_2 \\ x_1-x_2 \\ x_1+x_2 \end{bmatrix} = \begin{bmatrix} cx_1 \\ cx_2 \\ c(x_1-x_2) \\ c(x_1+x_2) \end{bmatrix} = \begin{bmatrix} cx_1 \\ cx_2 \\ cx_1-cx_2 \\ cx_1+cx_2 \end{bmatrix}$ So closed under scalar mult.

Thus, it is a subspace. basis: $\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \\ 1 \end{bmatrix} \right\}$.

Problem 4: a) False. By Rank Thearm, $\dim N(A) + \text{rank} = n$

b) False. Consider $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 4 \\ 0 \\ 0 \end{bmatrix} \right\}$.

c) False. The line must go through the origin.

d) True $2w - v$ is also a Lin combo of v_1, v_2, v_3 so it's a lin combo of $v_1, v_2, v_3, v_4, v_5, v_6$.

e) False $A = \begin{bmatrix} 1 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ is an example

Problem 5: ① $N(A) = \{0\}$
② $A(x) = b$ has a solution for every $b \in \mathbb{R}^n$ (ie, Span of cols of A is $\mathbb{R}^n$)
③ $\text{rref}(A) = I$.

Problem 6: $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ since want $\text{Span}\{a_1, a_2, a_3\} = \left\{ \begin{bmatrix} x \\ y \\ 0 \end{bmatrix} \;\middle|\; x, y \in \mathbb{R} \right\}$ and $N(A) = \left\{ t \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \;\middle|\; t \in \mathbb{R} \right\}$.

Challenge: Since $v_1, \ldots v_n$ span $\mathbb{R}^n$ and $b \in \mathbb{R}^n$, we know there are coefficients $c_1, \ldots, c_n$ such that $c_1v_1 + \cdots + c_nv_n = b$. Suppose there are also coefficients $d_1, \ldots, d_n$ such that $d_1v_1 + \cdots + d_nv_n = b$ too. Then $c_1v_1 + \cdots + c_nv_n = b = d_1v_1 + \cdots + d_nv_n$, so $(c_1 - d_1)v_1 + \cdots + (c_n - d_n)v_n = \vec{0}$. Since $v_1, \ldots v_n$ are lin indep, we must have $(c_i - d_i) = 0$ for all i, so $c_i = d_i$ for all i, hence unique Linear comb.