# Linear Algebra Exam Notes (Oct '22)

**Problem 1**: (a) $\left\{ \begin{bmatrix} -16 \\ 0 \\ 5 \\ 0 \end{bmatrix} + t \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + s \begin{bmatrix} 9 \\ 0 \\ 3 \\ 1 \end{bmatrix} \,\middle|\, t, s \in \mathbb{R} \right\}$

(b) basis for $C(A)$: $\left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 3 \\ 7 \\ 8 \end{bmatrix} \right\}$

**Problem 2**: $\begin{bmatrix} 1 & 3 & 2 & 2 \\ 2 & 7 & 6 & 8 \\ 3 & 9 & 6 & 7 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 3 & 2 & 2 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & 0 & 1 \end{bmatrix} \Rightarrow C(A) = \mathbb{R}^3$ since 3 pivots

a) So, since $C(A) = \mathbb{R}^3$, there is no $b \ni Ax = b$ has no solutions

b) No, cols are not L.I. $\begin{bmatrix} 2 \\ 6 \\ 6 \end{bmatrix}$ is a Linear combo of $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 3 \\ 7 \\ 9 \end{bmatrix}, \begin{bmatrix} 2 \\ 8 \\ 7 \end{bmatrix}$

**Problem 3**: $\begin{bmatrix} 1 & 2 & 3 \\ 1 & 1 & -2 \\ -1 & 0 & k \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 5 \\ 0 & 0 & 7-k \end{bmatrix}$ $\dim N(A) = 1$ when there is exactly one free variable, so ~~only~~ when $k \neq 7$

**Problem 4**: a) $\operatorname{Span}\left\{ \begin{bmatrix} 4 \\ 4 \\ 4 \\ 1 \\ 4 \end{bmatrix} \right\}$ b) By definition of span, $\vec{0} \in \operatorname{Span}\left\{ \begin{bmatrix} 4 \\ 4 \\ 1 \\ 4 \end{bmatrix} \right\}$ and the span is closed under addition and inverses.

**Problem 5**: $c_1 \begin{bmatrix} 2 & -4 & 2 \\ -4 & 8 & -4 \end{bmatrix} + c_2 \begin{bmatrix} 2 & -2 & -1 \\ 0 & 4 & 0 \end{bmatrix} + c_3 \begin{bmatrix} 1 & 0 & -2 \\ k & 0 & k \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ leads to

$$\left[\begin{array}{ccc|c} 2 & 2 & 1 & 0 \\ -4 & -2 & 0 & 0 \\ 2 & -1 & -2 & 0 \\ -4 & 0 & k & 0 \\ 8 & 4 & 0 & 0 \\ -4 & 0 & k & 0 \end{array}\right] \rightsquigarrow \left[\begin{array}{ccc|c} 2 & 2 & 1 & 0 \\ 0 & 2 & 2 & 0 \\ 0 & 3 & 3 & 0 \\ 0 & 4 & 2+k & 0 \\ 0 & 4 & 4 & 0 \\ 0 & 4 & 2+k & 0 \end{array}\right]$$

we need a pivot in each column, ~~and~~ when $2+k \neq 4$ we will have this. So for all $k \neq 2$.

**Problem 6**: $\begin{bmatrix} 1 & 1 & 3 & 1 \\ 2 & 2 & a & 1 \\ 0 & 0 & 4 & b \\ 0 & 0 & 2 & 6 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 1 & 3 & 1 \\ 0 & 0 & 6-a & 1 \\ 0 & 0 & 4 & b \\ 0 & 0 & 0 & b-12 \end{bmatrix}$

a) A will never be invertible because A will never have $\operatorname{rank} A = 4$

b) $\dim N(A) = 1$ when $\operatorname{rk} A = 3$ here, so if $b \neq 12$, a can be anything ~~if b=12, then~~ except $17/3$. So $\{(a, b) \mid a \neq 17/3 \text{ AND } b \neq 12\}$

c) i) The 4 column vectors ~~will never~~ form a LI set $\Leftrightarrow$ the matrix is invertible.

ii) $N(A) = \{0\} \Leftrightarrow$ the matrix is invertible.

## Problem 7

a) False. Suppose $v_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $v_2 = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$, $b = \begin{bmatrix} 5 \\ 0 \end{bmatrix}$ $v_3 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

b) False. Consider $B = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$ $C = \begin{bmatrix} 2 & 3 \\ 0 & 0 \end{bmatrix}$ $D = \begin{bmatrix} 2 & 3 \\ 1 & 1 \end{bmatrix}$

c) False. $Ax = 0$ always has the solution $\vec{x} = 0$.

d) True. Notice the matrix is square, and the invertible matrix thm applies.

(e) False. Consider $A = \begin{bmatrix} 1 & 2 \\ 1 & 2 \\ 1 & 3 \end{bmatrix}$.

## Challenge:

Prop: If $V$ is a vector space with basis $\{b_1 \ldots b_n\}$, then any vector in Span$\{b_1, \ldots b_n\}$ is a UNIQUE linear combination of these vectors.

Pf: Let $v \in$ Span$\{b_1 \ldots b_n\}$. Suppose

$$v = c_1 b_1 + c_2 b_2 + \cdots + c_n b_n \text{ and } v = d_1 b_1 + d_2 b_2 + \cdots + d_n b_n$$

for coefficients $c_1, \ldots, c_n, d_1, \ldots d_n$. Now,

$$v - v = \vec{0} \text{ and } v - v = (c_1 - d_1) b_1 + (c_2 - d_2) b_2 + \cdots + (c_n - d_n) b_n.$$

Since $\{b_1, \ldots b_n\}$ is a linearly independent set, by definition we have $c_1 - d_1 = 0$, $c_2 - d_2 = 0$, $\ldots$, $c_n - d_n = 0$, and this means $c_1 = d_1$, $c_2 = d_2$, $\ldots$ $c_n = d_n$. Thus, the two linear combinations ~~of~~ we started with are really the same, unique, linear combinations.