Remarks on Linear Algebra Exam (Oct '18 exam)

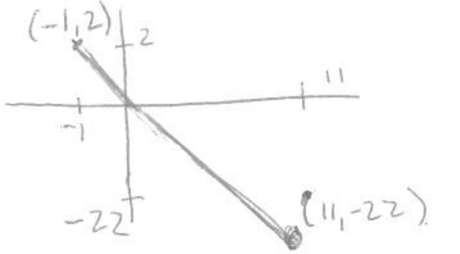
Problem 1: Associated matrix: $\begin{bmatrix} 2 & 0 & 1 & -1 \\ 0 & 4 & 1 & 3 \\ -1 & 6 & 1 & 5 \\ 3 & -2 & 1 & 0 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 2 & 0 & 1 & -1 \\ 0 & 4 & 1 & 3 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

a) Not 1-1 since $\ker T \neq 0$ ($\begin{bmatrix} -1/2 \\ -1/4 \\ 1 \\ 0 \end{bmatrix}$ is in $\ker T$ for ex)

b) Not onto by rank thm: $\dim N(A) + \dim C(A) = 4$ and $\dim N(A) > 0$.

c) basis $\ker T$ $\left\{ \begin{bmatrix} -2 \\ -1 \\ 4 \\ 0 \end{bmatrix} \right\}$ d) basis $\operatorname{Im} T$ $\left\{ \begin{bmatrix} 2 \\ 0 \\ -1 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 4 \\ 6 \\ -2 \end{bmatrix}, \begin{bmatrix} -1 \\ 3 \\ 5 \\ 0 \end{bmatrix} \right\}$

Problem 2:



Problem 3: $S = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \middle| \begin{matrix} a+2b=0 \\ c-3d=0 \end{matrix} \right\}$ is a subspace of $M_{2\times 2}$ since

- $0+2 0 = 0$ and $0 - 3 0 = 0$ so $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \in S$.
- if $\begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix}$ and $\begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix}$ are in S, then $a_1+2b_1=0$, $c_1-3d_1=0$, $a_2+2b_2=0$, $c_2-3d_2=0$

  and $\begin{bmatrix} a_1+a_2 & b_1+b_2 \\ c_1+c_2 & d_1+d_2 \end{bmatrix}$ satisfies $a_1+a_2+2(b_1+b_2) = a_1+2b_1+a_2+2b_2 = 0+0=0$
  $c_1+c_2-3(d_1+d_2) = c_1-3d_1+c_2-3d_2 = 0+0=0$

  so S is closed under addition.
- For $\begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix} \in S$ and $k \in \mathbb{R}$, $ka_1 + 2kb_1 = k(a_1+2b_1) = k \cdot 0 = 0$
  $kc_1 - 3kd_1 = k(c_1 - 3d_1) = k \cdot 0 = 0$

  so S is closed under scalar mult.

Problem 4: a) True $0^T = 0 \Rightarrow 0$ is in the set.
$(A+B)^T = A^T + B^T = -A - B = -(A+B)$ so closed under addition
$(cA)^T = c(A^T) = c(-A) = -cA$ so closed under scalar mult

b) False! Consider $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ $k=1$, $m=2$, $n=3$.

c) False! Consider $T(\vec{x}) = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} x$. Then $\left\{ T\begin{bmatrix} 0 \\ 3 \end{bmatrix}, T\begin{bmatrix} 3 \\ 0 \end{bmatrix} \right\}$ is lin dep

but $\left\{ \begin{bmatrix} 0 \\ 3 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \end{bmatrix} \right\}$ is an independent set.

d) True since the columns of $P_{\mathcal{B}}$ must be lin indep, and since $P_{\mathcal{B}}: V \to V$ it must be square.

Problem 5: $T: \mathbb{P}_4 \to \mathbb{P}_3$ $T(a_0 + a_1 t + a_2 t^2 + a_3 t^3 + a_4 t^4) =$

$$(3a_1 + 4a_2) + (6a_2 + 12a_3)t + (9a_3 + 12a_4)t^2 + 24a_4 t^3$$

a) $\operatorname{Ker} T$ basis: $\{1\}$ b) $\operatorname{Im} T$ basis $\{1, t, t^2, t^3\}$.

c) $(\mathbb{P}_4, \mathcal{B}) \xrightarrow{P_{\mathcal{B}}} (\mathbb{P}_4, \text{std}) \xrightarrow{A} (\mathbb{P}_3, \text{std}) \xrightarrow{P_{\gamma}^{-1}} (\mathbb{P}_3, \gamma)$

$$\begin{bmatrix} 0 & 0 & \frac{1}{5} & 0 \\ 1 & -2 & -\frac{1}{5} & 0 \\ 0 & 0 & 0 & \frac{1}{6} \\ 0 & 1 & -\frac{1}{5} & 0 \end{bmatrix} \begin{bmatrix} 0 & 3 & 4 & 0 & 0 \\ 0 & 0 & 6 & 12 & 0 \\ 0 & 0 & 0 & 9 & 24 \\ 0 & 0 & 0 & 0 & 12 \end{bmatrix} \begin{bmatrix} 3 & 0 & 1 & 1 & 1 \\ -4 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -2 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \end{bmatrix}$$

Challenge: Prove that if W is a subspace of $\mathbb{R}^n$, then $U = \{x \in \mathbb{R}^n \mid x^T w = 0 \ \forall w \in W\}$ is also a subspace of $\mathbb{R}^n$.

Pf: Notice that $\vec{0}^T w = 0$ for all vectors, so $\vec{0} \in U$.
If $\vec{x}, \vec{y} \in U$ and $c \in \mathbb{R}$, then $(c\vec{x} + \vec{y})^T w = c(\vec{x}^T + \vec{y}^T)w$
$= c\vec{x}^T \vec{w} + \vec{y}^T w = c \cdot 0 + 0 = 0$. Thus, U is closed under scalar multiplication and addition, and is a subspace.