Echelon Form:

Given a matrix A, there a 3 ways to change the matrix without changing the solution set. (If A is not augmented, then it's the solution set to $Ax = \vec{0}$ which is not changing.

These are the 3 elementary row operations on p66.

Identifying pivot variables and free variables helps us write down solution sets:

Ex:
$$\left[\begin{array}{cccccc|c} 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{array}\right]$$

pivot x_1 pivot x_3 pivot x_5

Here, x_2, x_4, x_6 are free. Recall, a solution is a set of vectors $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix}$ which satisfies the system.... hence my labelling for names of pivot & free var.

x_6 can be anything
$x_5 = 0$
x_4 can be anything
$x_3 = -x_6$
x_2 can be anything
$x_1 = -x_4$

So! Every time you choose 3 numbers (something for x_2, x_4 & x_6), you get a solution. Say $x_2 = 1$, $x_4 = 2$, $x_6 = 3$

then $\begin{bmatrix} -2 \\ 1 \\ -3 \\ 2 \\ 0 \\ 3 \end{bmatrix}$ is a solution. Check!

Let's do this smartly:

(1) Choose $x_2 = 1$, $x_4 = 0$, $x_6 = 0$. Sol: $\begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$

(2) Choose $x_2 = 0$, $x_4 = 1$, $x_6 = 0$ Sol: $\begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$

(3) Choose $x_2 = 0$ $x_4 = 0$ $x_6 = 1$ Sol $\begin{bmatrix} 0 \\ 0 \\ -1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$

Full set of solutions:

$$\left\{ c_1 \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + c_3 \begin{bmatrix} 0 \\ 0 \\ -1 \\ 0 \\ 0 \\ 1 \end{bmatrix} \;\middle|\; c_1, c_2, c_3 \text{ are real \#s} \right\}$$

This works nicely since $0 + 0 + 0 = 0$.

Yikes! we DO have to be more careful when we are solving $Ax = b$ if $b \neq 0$.

So... if our system had been

$$\left[\begin{array}{cccccc|c} 1 & 0 & 0 & 1 & 0 & 0 & 2 \\ 0 & 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 5 \end{array}\right]$$

Then, first we need to find a solution. Let's choose all 3 free variables as 0.

$x_2 = 0$
$x_4 = 0$
$x_6 = 0$

$$\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$ and now finish solving for x_5, x_3, x_1

$$\begin{bmatrix} 2 \\ 0 \\ 1 \\ 0 \\ 5 \\ 0 \end{bmatrix}.$$

Full set of solutions to $Ax = \begin{bmatrix} 2 \\ 1 \\ 5 \end{bmatrix}$.

$$\left\{ \begin{bmatrix} 2 \\ 0 \\ 1 \\ 0 \\ 5 \\ 0 \end{bmatrix} + c_1 \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + c_3 \begin{bmatrix} 0 \\ 0 \\ -1 \\ 0 \\ 0 \\ 1 \end{bmatrix} \;\middle|\; c_1, c_2, c_3 \text{ are real \#s.} \right\}.$$