C(A) basis, N(A) basis

If A is an $m \times n$ matrix, we can associate subspaces to it – one a subspace of $\mathbb{R}^m$ and one a subspace of $\mathbb{R}^n$.

$C(A)$, the column space is a subspace of $\mathbb{R}^m$. By definition $C(A) = \text{Span}\{v_1, \ldots v_n\}$ where the v_i vectors are the columns of A.

Ex: $A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 6 & 8 \end{bmatrix}$

$$C(A) = \text{Span}\left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 4 \end{bmatrix}, \begin{bmatrix} 3 \\ 6 \end{bmatrix}, \begin{bmatrix} 4 \\ 8 \end{bmatrix} \right\}$$

The set of vectors $\left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 4 \end{bmatrix}, \begin{bmatrix} 3 \\ 6 \end{bmatrix}, \begin{bmatrix} 4 \\ 8 \end{bmatrix} \right\}$ clearly Span $C(A)$ (we made $C(A)$ this way!!) But are these vectors a <u>basis</u> for $C(A)$?

No! They are not linearly independent. We need to find a largest subset which is also a lin indep set.

To do this, we compute the echelon form of A:
$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 6 & 8 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Every column of A with a pivot should be part of our basis. BEWARE! We need those columns from $\underline{\underline{A}}$ not ~~the~~ echelon form

basis for $C(A)$: $\left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right\}$.

$N(A)$, the null space, is a subspace of $\mathbb{R}^4$.
By definition, $N(A) = \{\vec{x} \in \mathbb{R}^n \mid A\vec{x} = \vec{0}\}$.

Yes! We've done this before! Each free variable in the echelon form contributes a vector to the basis. And we won't need any others.

Notice, it's called the nullspace of A. That's because A "makes null" or "kills" all the vectors in it.

Now, those basic solutions - the ones we got by singling out just one free variable at a time - we used those so the vectors would be lin indep.

Let's see: $A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 6 & 8 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$.

x_2, x_3, x_4 are free. When we write the sol set to $Ax = 0$ parametrically, we get:

$$N(A) = \left\{ x_2 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -3 \\ 0 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -4 \\ 0 \\ 0 \\ 1 \end{bmatrix} \;\middle|\; x_2, x_3, x_4 \in \mathbb{R} \right\}.$$

and a basis for $N(A)$ is $\left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -4 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$.

These 3 vectors span $N(A)$ since every vector in $N(A)$ is a linear combination of them.

They are linearly independent too -

if $c_1 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} -3 \\ 0 \\ 1 \\ 0 \end{bmatrix} + c_3 \begin{bmatrix} -4 \\ 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$

then to get the 2^{nd} coord to sum to 0, c_1 MUST $= 0$.
to get the 3^{rd} coord to sum to 0, c_2 MUST $= 0$
to get the 4^{th} coord to sum to 0 c_3 MUST $= 0$.
Independence!