Diagonalization

... A Basis of Eigenvectors

Let $T: \mathbb{R}^n \to \mathbb{R}^n$ be a linear transformation.

We know
- when we describe the action of T on a basis of $\mathbb{R}^n$, we have enough information to know ~~what~~ how T acts on every vector in $\mathbb{R}^n$.
- we can describe T by $T(x) = Ax$ where A is an $n \times n$ std matrix. (1st col will be $T\begin{pmatrix}1\\0\\ \vdots\\0\end{pmatrix}$ 2nd col $T\begin{pmatrix}0\\1\\ \vdots\\0\end{pmatrix}$ etc).

But not all descriptions of functions are created equally! Consider "the function from $\mathbb{R}$ to $\mathbb{R}$ which, upon receiving an input, doubles that input, then removes three units then reduces the result by one fourth" vs
"$f: \mathbb{R} \to \mathbb{R}$, $f(x) = \dfrac{2x-3}{4}$."

For some transformations, we can find a basis of $\mathbb{R}^n$ that makes the description of that transformation very nice.

Given $T(x) = Ax$, $(T: \mathbb{R}^n \to \mathbb{R}^n)$ if we can find

n linearly independent eigenvectors of A,

they will form a basis for $\mathbb{R}^n$, and we can diagonalize A.

That is, instead of the standard language - ie, the standard basis vectors - if we use this eigenvector language - this basis of eigenvectors - the description of T will be short & sweet!

Ex: $T: \mathbb{R}^3 \to \mathbb{R}^3$ given by $T(x) = \begin{bmatrix} 1 & 3 & 3 \\ -3 & -5 & -3 \\ 3 & 3 & 1 \end{bmatrix} x$

You can check... do it!.. that

$\lambda = 1$ is an eigenvalue with $E_1 = \text{Span}\left\{\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}\right\}$

$\lambda = -2$ is an eigenvalue with $E_{-2} = \text{Span}\left\{\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}\right\}$.

Let $\mathcal{B} = \left\{\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}\right\}$

You should be able to check that $\mathcal{B}$ is a basis for $\mathbb{R}^3$.

Now... "magically"...

$$\underbrace{\begin{bmatrix} 1 & 3 & 3 \\ -3 & -5 & -3 \\ 3 & 3 & 1 \end{bmatrix}}_{A} = \begin{bmatrix} 1 & -1 & -1 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{bmatrix} \begin{bmatrix} 1 & -1 & -1 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}^{-1}$$

$$= \underbrace{\begin{bmatrix} 1 & -1 & -1 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}}_{P} \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{bmatrix}}_{D} \underbrace{\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 1 \\ -1 & -1 & 0 \end{bmatrix}}_{P^{-1}}$$

$P^{-1}AP \quad = D$