Finding Eigenvectors & Eigenvalues.

An eigenvector is a non-zero vector x so that $Ax = \lambda x$ for some scalar λ. (that's it's traditional name - don't blame me for the choice).

- A has to be a square matrix
- x has to be non-zero
- each eigen-vector is associated to an eigen-value, - just one - and a matrix can have several different eigen-values.

So how do we find them?

We do algebra with matrices, as you would have done it (without matrices) in elementary school.

$$Ax = \lambda x$$

we don't know x!
we don't know λ!

$$Ax - \lambda x = \vec{0}$$

$$\text{"}(A-\lambda)x = \vec{0}\text{"}$$

this doesn't make sense yet since

so "recognize" that
$\lambda x = \lambda I x$

we have $(A - \lambda I)x = \vec{0}$.

Hey! we <u>know</u> how to find

solutions to $(A-\lambda I)x = \vec{0}$ and

<u>when</u> they happen! (Remember,

we want (non-zero) non-trivial vectors $\vec{x}$.)

So when $\det(A-\lambda I) = 0$ the matrix $A-\lambda I$ is NOT invertible, and for these values of λ that make $\det(A-\lambda I) = 0$, we'll find eigen-vectors!

$\det(A-\lambda I)$ is called the characteristic polynomial

... because it characterizes what happens with eigenvectors for the original matrix A.

Eigenvector Example

$$A = \begin{bmatrix} 4 & 3 & 2 \\ 0 & 0 & 1 \\ 0 & 0 & 4 \end{bmatrix}.$$ To find eigenvalues and eigenvectors.

Step 1: Form $A-\lambda I$ and find the determinant:

$$A-\lambda I = \begin{bmatrix} 4-\lambda & 3 & 2 \\ 0 & -\lambda & 1 \\ 0 & 0 & -\lambda \end{bmatrix}$$

$$\det(A-\lambda I) = (4-\lambda)\left[(-\lambda)^2\right] - 3\left[0\right) + 2(0)$$
$$= (4-\lambda)(\lambda^2)$$

Step 2: Find the roots of this polynomial:
$(4-\lambda)\lambda^2 = 0$ when $\lambda = 4, \lambda = 0$.

Step 3: For each of these λ's find $N(A-\lambda I)$.

$\lambda = 4$: $A-4I = \begin{bmatrix} 0 & 3 & 2 \\ 0 & -4 & 1 \\ 0 & 0 & -4 \end{bmatrix}$

pivot pivot.

hey! free variable is here

$$N(A-4I) = \text{Span}\left\{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right\}$$

check: $\begin{bmatrix} 4 & 3 & 2 \\ 0 & 0 & 1 \\ 0 & 0 & 4 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 4 \\ 0 \\ 0 \end{bmatrix} = 4\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$:)