A <u>Linear Combination</u> of vectors $v_1, \dots, v_k$

is any expression of the form

$c_1 v_1 + \cdots + c_k v_k$ where the c_i's are real #s.

The <u>Span</u> of vectors $v_1, \dots, v_k$ is

the set of <u>all</u> linear combinations of

these vectors.

Ex: $\begin{bmatrix} -2 \\ 1 \\ -3 \\ 2 \\ 5 \\ 3 \end{bmatrix}$ is a Linear Combination of $\begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 5 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \\ 5 \\ 0 \end{bmatrix}$ & $\begin{bmatrix} 0 \\ 0 \\ -1 \\ 0 \\ 5 \\ 1 \end{bmatrix}$

since $\begin{bmatrix} -2 \\ 1 \\ -3 \\ 2 \\ 5 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 5 \\ 0 \end{bmatrix} + 2\begin{bmatrix} -2 \\ 0 \\ 0 \\ 1 \\ 5 \\ 0 \end{bmatrix} + 3\begin{bmatrix} 0 \\ 0 \\ -1 \\ 0 \\ 5 \\ 1 \end{bmatrix}$

This is equivalent to saying

$\begin{bmatrix} -2 \\ 1 \\ -3 \\ 2 \\ 5 \\ 8 \end{bmatrix}$ is in $\text{Span}\left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 5 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \\ 5 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ -1 \\ 0 \\ 5 \\ 1 \end{bmatrix} \right\}$.

* Lots of our work this term will be saying the same thing in different language... and <u>recognizing</u> this is our challenge.

(1) It's not always obvious if a vector is in the span of certain given vectors.

Is $\begin{bmatrix} 2 \\ 4 \\ -3 \end{bmatrix}$ in Span$\left\{ \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} -3 \\ -3 \\ -6 \end{bmatrix}, \begin{bmatrix} 4 \\ 0 \\ 8 \end{bmatrix} \right\}$?

This is equivalent to asking are there real #s c_1, c_2, c_3 so that

$$c_1 \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} + c_2 \begin{bmatrix} -3 \\ -3 \\ -6 \end{bmatrix} + c_3 \begin{bmatrix} 4 \\ 0 \\ 8 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \\ -3 \end{bmatrix}$$

which is equivalent to asking if

$$\begin{bmatrix} 1 & -3 & 4 \\ 1 & -3 & 0 \\ 2 & -6 & 8 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \\ -3 \end{bmatrix}$$ is a consistent system. ~~of~~

And now you can find the echelon form of this augmented Matrix and decide...

NO. NOT in the Span.

(2) Span$\left\{ \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} -3 \\ -3 \\ -6 \end{bmatrix}, \begin{bmatrix} 4 \\ 0 \\ 8 \end{bmatrix} \right\}$ = Span$\left\{ \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 4 \\ 0 \\ 8 \end{bmatrix} \right\}$

= Span$\left\{ \begin{bmatrix} 5 \\ 1 \\ 10 \end{bmatrix}, \begin{bmatrix} 4 \\ 0 \\ 8 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 4 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 6 \\ 1 \\ 12 \end{bmatrix} \right\}$

... and lots of others ...

Linear Independence

A set $\{v_1, \ldots v_k\}$ of vectors is <u>linearly independent</u> if and only if the <u>only</u> solution to the linear combo

$$c_1 v_1 + c_2 v_2 + \ldots + c_k v_k = \vec{0}$$

is $c_1 = c_2 = c_3 = \ldots = c_k = 0$.

* So, a set is linearly independent if there is nothing redundant in the set - every vector is crucial.

* Span, on the other hand, is a way of describing all the vectors one can reach.

<u>Ex</u>: $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}$ is a linearly independent set in $\mathbb{R}^3$.

2 vectors in this set

$\text{Span} \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}$ is a whole plane of vectors in $\mathbb{R}^3$

$$= \left\{ \begin{bmatrix} x \\ y \\ 0 \end{bmatrix} \middle|\ x, y \text{ are real } \# \right\}$$

Adding another vector to the original set

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}$$ might or might not change the linear independence.

If it does change the independence, (so the set is dependent) then it does NOT change the Span.

If it does NOT change the independence, then it does change the Span!

Ex: $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \\ 0 \end{bmatrix} \right\}$ linearly dependent set

$$\text{Span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \\ 0 \end{bmatrix} \right\} = \text{Span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}$$

Ex: $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix} \right\}$ linearly independent set.

$\text{Span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix} \right\}$ is all of $\mathbb{R}^3$!