LA Practice Final Exam Notes (Nov '22 ex)

Problem 1: $\begin{bmatrix} 2 & 3 & 6 & 4 \\ 1 & 1 & 2 & h \\ 2 & 2 & -1 & 2h \\ 4 & 5 & 5 & 3h \end{bmatrix} \rightsquigarrow \begin{bmatrix} 2 & 3 & 6 & 4 \\ 0 & 1 & 2 & 4-2h \\ 0 & 0 & 5 & 0 \\ 0 & 0 & 0 & 4-h \end{bmatrix}$

(a) So: There will always be at least 3 pivots, so $\dim N(A)$ cannot be 2: it can only be 1 or 0.

(b) When $h=4$ then $\dim N(A)=1$ since only 3 pivots, and if $h \neq 4$, then A is invertible.

Problem 2 Rewriting, $\left[\begin{array}{ccc|c} 1 & -1 & 0 & 3 \\ 0 & 1 & -2 & -3 \\ 1 & 0 & -2 & 0 \end{array}\right] \rightsquigarrow \left[\begin{array}{ccc|c} 1 & -1 & 0 & 3 \\ 0 & 1 & -2 & -3 \\ 0 & 1 & -2 & -3 \end{array}\right] \rightsquigarrow \left[\begin{array}{ccc|c} 1 & -1 & 0 & 3 \\ 0 & 1 & -2 & -3 \\ 0 & 0 & 0 & 0 \end{array}\right]$

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ -3 \\ 0 \end{bmatrix} + t \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix} \text{ for any real \# } t.$$

Problem 3 $A = \begin{bmatrix} -1 & 3 & -1 & 0 \\ 2 & 4 & 2 & 10 \\ -3 & 6 & -3 & -3 \end{bmatrix}$ (b) basis for $C(A)$ $\left\{ \begin{bmatrix} -1 \\ 2 \\ -3 \end{bmatrix}, \begin{bmatrix} 3 \\ 4 \\ 6 \end{bmatrix} \right\}$

Problem 4

a) Using standard coordinates the $\begin{bmatrix} 1 & 4 & -1 \\ 2 & 5 & 4 \\ 3 & 6 & 2 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 4 & -1 \\ 0 & 3 & -6 \\ 0 & 0 & 7 \end{bmatrix}$.

Since there is a pivot in each col, the vectors are linearly independent. Since $\mathbb{P}_2$ has dim 3, and we have 3 lin indep vectors, this is a basis.

b) $\begin{bmatrix} 1 & 2 & 1 & 0 \\ 1 & 0 & 1 & 1 \\ 0 & 0 & -1 & 1 \\ 1 & 3 & 2 & 0 \end{bmatrix}^{-1} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 10 \end{bmatrix} \begin{bmatrix} 1 & 4 & -1 \\ 2 & 5 & 4 \\ 3 & 6 & 2 \end{bmatrix}$

Problem 5 a) False $A = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$ $v_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ $v_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ $A\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$.

b) True by Invertible Matrix Theorem.

c) True Since cols are Lin indep, there will be n pivots, so this is the rank of the Matrix, ie, $\dim C(A)$.

d). False S does not include the $\vec{0}$ vector, for example

e) False The line given by $y = 3x+1$ does not include the $\vec{0}$ vector. ie the line $\left\{ \begin{bmatrix} x \\ y \end{bmatrix} \middle| \, y = 3x+1 \right\}$ is not a subspace of $\mathbb{R}^2$

f). True. Any span is a subspace

Problem 6 $\operatorname{Ker} T = \left\{ \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \middle| \, a_{21} = a_{12} \text{ and } a_{11} = 3a_{22} \right\}$

a) so, a basis is $\left\{ \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix} \right\}$.

b). Since $\dim M_{2\times2} = 4$ and $\dim \operatorname{Ker} T = 2$, $\dim \operatorname{Im} T = 2 \neq \dim \mathbb{P}^2$, hence no, not onto.

Problem 7: a) Rank Thm says if A is an $m \times n$ matrix, $\dim N(A) + \dim C(A) = n$.

b). If A is invertible, $Ax = b$ has a unique solution for each b, that sol given by $x = A^{-1}b$.

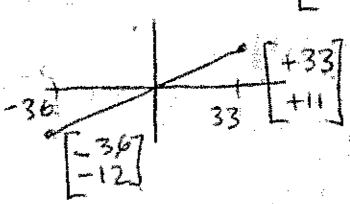
c) An eigenvector for a matrix A is a non-zero vector v such that $Av = \lambda v$ for some scalar λ.

Problem 8 $\det(A-\lambda I) = (2-\lambda)(-\lambda(1-\lambda)-2) = (2-\lambda)(\lambda-2)(\lambda+1)$.

$E_2 = \operatorname{Span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} \right\}$ $E_{-1} = \operatorname{Span}\left\{ \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} \right\}$ so.

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 2 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 2 & 1 \end{bmatrix}^{-1}$$

Problem 9



Consider $T\begin{pmatrix} 1 \\ 0 \end{pmatrix}, T\begin{pmatrix} 0 \\ 0 \end{pmatrix}$ etc.

Challenge: Suppose $T: V \to W$ is a linear transformation, and there is a non-zero vector v in $\operatorname{Ker} T$. Then $T(v) = 0$ and since T is linear, $T(0) = 0$ too, so T is NOT 1-1. Thus, if T is 1-1, then $\operatorname{Ker} T = \{0\}$.