Math 240 - Linear Algebra
November 20, 2022 - Final Exam

DIRECTIONS: Make sure to show all relevant work in a clear and complete fashion. You must show enough work to justify your answer! You may do the problems in any order in your blue book, but be sure to do all parts of a given problem are located together. There are 83 points on this exam. Good luck!

Problem 1. (8 pts) Let $\mathbf{A}$ be the matrix

$$\begin{bmatrix} 2 & 3 & 6 & 4 \\ 1 & 1 & 2 & h \\ 2 & 2 & -1 & 2h \\ 4 & 5 & 5 & 3h \end{bmatrix}.$$

(a) Find all values of h such that the null space of $\mathbf{A}$ has dimension 2. Explain.

(b) Find all values of h such that $\mathbf{A}$ is invertible. Explain.

Problem 2. (8 pts) Find all the triples of points a, b, c, if any, so that

$$\begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} \text{ is a solution to } \begin{bmatrix} a & b & -3 \\ 2 & -b & -c \\ a & 3 & -c \end{bmatrix} \mathbf{x} = \begin{bmatrix} -3 \\ -1 \\ -3 \end{bmatrix}.$$

Problem 3. (6 pts) Let $T : \mathbb{R}^4 \to \mathbb{R}^3$ be the linear transformation defined by $T\left(\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}\right) = \begin{bmatrix} 3b + 2d - a - c \\ 2a + 4b + 2c + 10d \\ 6b - 3a - 3c - 3d \end{bmatrix}$.

(a) Find the standard matrix representation of T.

(b) Find a basis for the column space of the matrix you found.

Problem 4. (10 pts) Let $\mathbb{P}_n$ be the vector space of polynomials of degree less than or equal to n. Consider $T : \mathbb{P}_2 \to \mathbb{P}_3$ defined by $T(a_0 + a_1x + a_2x^2) = 5a_1x^2 + 10a_2x^3$.

(a) Show that $\beta = \{1 + 2x + 3x^2, 4 + 5x + 6x^2, -1 + 4x + 2x^2\}$ is a basis for $\mathbb{P}_2$.

(b) Find a matrix or product of matrices representing $T : (\mathbb{P}_2, \beta) \to (\mathbb{P}_3, C)$ where $C = \{1 + x + x^3, 2 + 3x^3, 1 + x - x^2 + 2x^3, x + x^2\}$.

Problem 5. (18 pts) True or False? For each statement below, decide whether it is true or false. If it is true, briefly explain why. If it is false, cite a definition, theorem or calculation that shows the statement to be false.

(a) If $\mathbf{v_1}$ and $\mathbf{v_2}$ are eigenvectors for a matrix A, then $\mathbf{v_1} + \mathbf{v_2}$ is an eigenvector for $\mathbf{A}$.

(b) If $\mathbf{A}$ is an invertible matrix, then the linear transformation given by $T(\mathbf{x}) = \mathbf{Ax}$ is 1-1.

(c) If the columns of an $m \times n$ matrix $\mathbf{A}$ are linearly independent, then the column space of $\mathbf{A}$ is dimension n.

(d) The set $S = \{\mathbf{A} | \, det(\mathbf{A}) = 1\}$ is a subspace of $M_{n \times n}$.

(e) Any line in $\mathbb{R}^2$ is a subspace of $\mathbb{R}^2$. (That is, given a line, the set of vectors whose heads lie on that line is a subspace.)

(f) $Span\{\sin x, \sin(2x) + 1\}$ is a subspace of the vector space of functions from $\mathbb{R}$ to $\mathbb{R}$.

Problem 6. (8 pts) Let T be the linear transformation $T : M_{2\times 2} \to \mathbb{P}^2$ defined by

$$T\left(\begin{bmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{bmatrix}\right) = (3a_{2,2} - a_{1,1}) + (a_{1,2} - a_{2,1})t + (a_{2,1} - a_{1,2})t^2.$$

(a) Find a basis for $Ker(T)$. Be sure to give your final answer using vectors in the vector space, not coordinates.

(b) Is this transformation onto? Explain.

Problem 7. (9 pts) Give brief answers to each of the following.

(a) What does the rank theorem say?

(b) When is a matrix invertible, and how does this relate to solving systems of equations?

(c) Give a formal definition of the term eigenvector for a matrix.

Problem 8. (8 pts) Diagonalize the matrix or explain why it can't be done. (Remember, factored form is nice here...so don't multiply things out too much!) Recall that diagonalizing a matrix involves a product of matrices, although if this can be done, you do not need to compute an inverse.

$$\begin{bmatrix} 2 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 2 & 1 \end{bmatrix}$$

Problem 9. (8 pts) Consider the picture of the house below. Draw the image of the house under the transformation

$$T(\mathbf{x}) = \begin{bmatrix} 3 & -6 \\ 1 & -2 \end{bmatrix}\mathbf{x}$$

Briefly explain your reasoning.