Linear transformations are <u>functions</u>

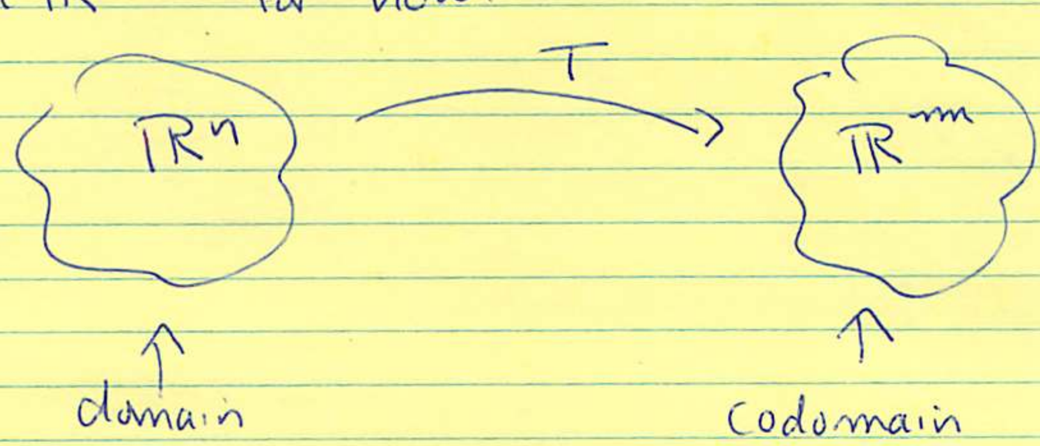
between vector spaces that respect linear combinations. That is, ~~they~~ we can do algebra in the domain, then apply the function, or apply the function to each piece, then do the algebra in the codomain,

and either way, we get the same answer!

In Calculus you studied $f: \mathbb{R} \to \mathbb{R}$

Here we study $T: V \longrightarrow W$ where

V, W are vector spaces. Think $\mathbb{R}^n$ and $\mathbb{R}^m$ for now.



$T(u+v) = T(u) + T(v)$ for all u, v in domain

and $T(cu) = cT(u)$ for all u, all real #s c.

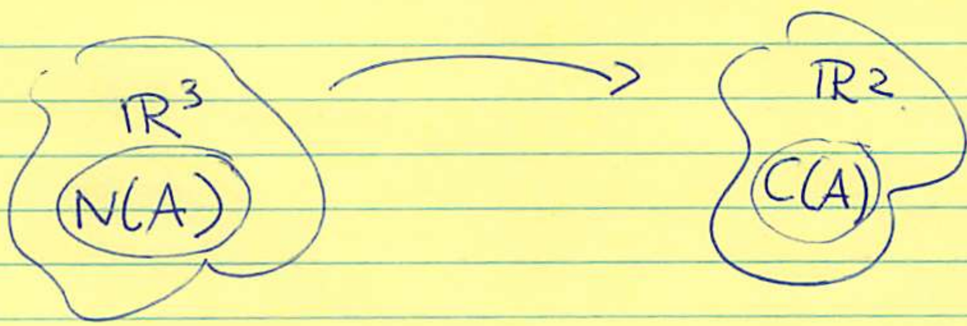
<u>Prime Example</u>: Let A be an $m \times n$ matrix.

$T(\vec{x}) = A\vec{x}$ is a linear transformation

Hey! We know a lot about functions of this form. We just haven't been thinking about them as functions.

Ex $A = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 2 & 5 \end{bmatrix}$

$$A \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 + x_2 + 2x_3 \\ 2x_1 + 2x_2 + 5x_3 \end{bmatrix}.$$



Bases and Linear Transformations

"Remember" that if we have a basis for a vector space, every vector in the vector space can be written uniquely as a linear combination of the basis vectors.

Ex $\beta = \left\{ \begin{bmatrix} 3 \\ 2 \end{bmatrix}, \begin{bmatrix} -1 \\ 4 \end{bmatrix} \right\}$ is a basis for $\mathbb{R}^2$.

So! If we know what a linear transformation does on those two vectors, we know what it transforms every vector to!

Ex $v = \begin{bmatrix} 1 \\ 10 \end{bmatrix} = 1 \begin{bmatrix} 3 \\ 2 \end{bmatrix} + 2 \begin{bmatrix} -1 \\ 4 \end{bmatrix}$

so if $T \begin{bmatrix} 3 \\ 2 \end{bmatrix} = \begin{bmatrix} 5 \\ 0 \\ 9 \\ -1 \end{bmatrix}$ and $T \begin{bmatrix} -1 \\ 4 \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \\ -3 \\ 5 \end{bmatrix}$

then
$$T\left(\begin{bmatrix} 1 \\ 10 \end{bmatrix} \right) = T\left(1 \begin{bmatrix} 3 \\ 2 \end{bmatrix} + 2 \begin{bmatrix} -1 \\ 4 \end{bmatrix} \right)$$
$$= 1\, T \begin{bmatrix} 3 \\ 2 \end{bmatrix} + 2\, T \begin{bmatrix} -1 \\ 4 \end{bmatrix}$$
$$= \begin{bmatrix} 5 \\ 0 \\ 9 \\ -1 \end{bmatrix} + 2 \begin{bmatrix} 3 \\ 0 \\ -3 \\ 5 \end{bmatrix} = \begin{bmatrix} 11 \\ 0 \\ 3 \\ 9 \end{bmatrix}.$$

(By the way, we say the β-coordinates of v are $\begin{bmatrix} 1 \\ 2 \end{bmatrix}_\beta$)

Now, you can check that for this ~~bases~~ example,

$$\begin{bmatrix}1\\0\end{bmatrix} = \frac{2}{7}\begin{bmatrix}3\\2\end{bmatrix} - \frac{1}{7}\begin{bmatrix}-1\\4\end{bmatrix}$$

$$\begin{bmatrix}0\\1\end{bmatrix} = \frac{1}{14}\begin{bmatrix}3\\2\end{bmatrix} + \frac{3}{14}\begin{bmatrix}-1\\4\end{bmatrix}$$

So $T\begin{bmatrix}1\\0\end{bmatrix} = T\left(\frac{2}{7}\begin{bmatrix}3\\2\end{bmatrix} - \frac{1}{7}\begin{bmatrix}-1\\4\end{bmatrix}\right) = \frac{2}{7}T\begin{bmatrix}3\\2\end{bmatrix} - \frac{1}{7}T\begin{bmatrix}-1\\4\end{bmatrix}$

$$= \frac{2}{7}\begin{bmatrix}5\\0\\9\\-1\end{bmatrix} - \frac{1}{7}\begin{bmatrix}3\\0\\-3\\5\end{bmatrix} = \begin{bmatrix}1\\0\\3\\-1\end{bmatrix}$$

and $T\begin{bmatrix}0\\1\end{bmatrix} = T\left(\frac{1}{14}\begin{bmatrix}3\\2\end{bmatrix} + \frac{3}{14}\begin{bmatrix}-1\\4\end{bmatrix}\right) = \frac{1}{14}T\left(\begin{bmatrix}3\\2\end{bmatrix}\right) + \frac{3}{14}T\left(\begin{bmatrix}-1\\4\end{bmatrix}\right)$

$$= \frac{1}{14}\begin{bmatrix}5\\0\\9\\-1\end{bmatrix} + \frac{3}{14}\begin{bmatrix}3\\0\\-3\\5\end{bmatrix} = \begin{bmatrix}1\\0\\0\\1\end{bmatrix}.$$

Now it's easyer to see what T does to each vector because $\begin{bmatrix}x_1\\x_2\end{bmatrix} = x_1\begin{bmatrix}1\\0\end{bmatrix} + x_2\begin{bmatrix}0\\1\end{bmatrix}$!

$$T\left(\begin{bmatrix}x_1\\x_2\end{bmatrix}\right) = x_1\begin{bmatrix}1\\0\\3\\-1\end{bmatrix} + x_2\begin{bmatrix}1\\0\\0\\1\end{bmatrix} = \begin{bmatrix}x_1+x_2\\0\\3x_1\\-x_1+x_2\end{bmatrix}.$$

$$= \underbrace{\begin{bmatrix}1&1\\0&0\\3&0\\-1&1\end{bmatrix}}_{\text{Standard Matrix of } T!}\begin{bmatrix}x_1\\x_2\end{bmatrix}$$